8

Ratios

Ratios are part of a large web of mathematical concepts and skills known as *proportional reasoning* that make use of ideas from multiplication, division, fractions, and measurement. Proportional reasoning is the ability to make and use multiplicative comparisons among quantities. These comparisons are expressed as ratios and rates. We use ratios and rates every day to convey information: the car is traveling at 25 mph, place 2 roses and 3 pieces of greenery in every bouquet, ground beef costs $1.89 per pound, 3 out of every 8 doctors majored in biochemistry, and there is a 30% chance of thunderstorms.

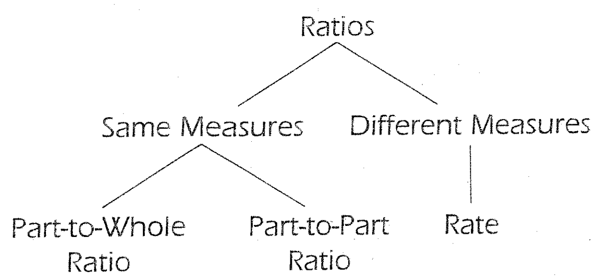
Proportionality is a complex topic; it is estimated that over half the adult population do not reason proportionally (Lamon 1999). Furthermore, students' understanding of this topic takes years to develop—a two-week unit on ratios in sixth grade is not enough for most students to acquire anything but a superficial ability to apply an algorithm. Instead, it is necessary for students informally to explore ideas related to thinking multiplicatively throughout the elementary grades and spend significant time developing these concepts in middle school.

1. What Are Ratios?

A *ratio* is a comparison between two or more quantities. The quantities can be either numbers or measurements. Comparisons such as 5 pencils for $1.09, 12 degrees per hour, 4 girls for every 5 boys, $\frac{1}{2}$ cup lemonade concentrate to 4 cups of water, and 3 red marbles compared with 8 marbles altogether are all ratios. Sometimes ratios are used to compare more than two quantities; 1 part vinegar, 1 part linseed oil, and 1 part turpentine is a ratio for an old-fashion solution used to refinish antique furniture.

Ratios are further classified based on the type of comparison. If we are comparing measures of the same type, such as people, inches, and marbles, we can either make part-to-whole comparisons or part-to-part comparisons. Part-to-whole ratios can be interpreted as fractions because they compare a part with a whole (e.g., the ratio 7 to 20, when comparing 7 girls with a total of 20 students in a classroom, is a fraction that tells us what part of the set—$\frac{7}{20}$—are girls). Other ratios compare parts of a set to other parts of a set. For example, we can compare 2 blue marbles with 6 red marbles in a set of marbles and then express the ratio of blue marbles to red marbles as 2:6, or 1:3. Starting with either a part-to-whole ratio or a part-to-part ratio, we are able to construct other ratios that apply. For example, if we use the 2 blue marbles to 6 red marbles ratio, the ratios of blue marbles to total number of marbles (2:8, part-whole ratio); red marbles to total number of marbles (6:8, part-whole ratio); and red marbles to blue marbles (6:2, part-part ratio) all provide information about the relationship between the two colors of marbles.

When two different types of measures are being compared, the ratio is usually called a *rate*. Comparisons involving number of miles and number of hours, number of dollars and bags of sugar, and number of minutes and number of feet are all rates. Rates sometimes involve a comparison with 1 (60 miles per 1 hour, 2 bags for 1 dollar, 0.25 feet in 1 minute); these "unit" rates are easier for us to generalize and extend (if I can drive 60 miles in 1 hour, my trip of 180 miles will take about 3 hours). Everyday usage of the terms *ratio* and *rate* is not always precise or correct. For example, rates are often simply referred to as ratios (which is correct but not very precise). On the other hand, some relationships are incorrectly called rates—they are not actually rates—since the measures are the same: birth rates compare number of people born with a designated number of people (usually 1000) and while this comparison is a ratio, it technically isn't a rate. A diagram may help you make sense of the different types of ratios.



There are two common notations used to identify ratios: the colon notation and the fraction notation. While interchangeable, the choice of notation helps focus our attention on different aspects of interpretation. The colon notation is most often used with part-to-part comparisons (3 adults compared with 8 children, 3:8), while the fraction notation is favored with part-to-whole comparisons (3 adults compared with 11 people total, $\frac{3}{11}$). When fraction notation is used to compare two parts, it helps if the parts are labeled: $\frac{3\ \text{adults}}{8\ \text{children}}$. Otherwise, devoid of context, the conceptual differences between ratios and fractions can be lost and symbols can be interpreted in unexpected ways (you might think $\frac{3}{8}$ represents 3 out of a total of 8). Finally, fraction notation is commonly used when computing with ratios (e.g., finding equivalent ratios or finding the value of an unknown in a proportion) regardless of the type of relationship between the numbers.

Types of Comparisons

The concept of comparison or change between quantities is both simple and quite complex. Children start comparing quantities at a very young age—they notice a friend has more matchbox cars, they want a smaller helping of mashed potatoes at dinner, they wonder why there are fewer girls playing basketball than playing soccer. As teachers we regularly ask comparison questions: *If there are 8 boys in grade 1 and 7 boys in grade 2, are there more boys in the first or the second grade? How many more* *students have brown eyes than blue eyes? The temperature dropped how many degrees yesterday?* To answer these comparison questions, in which we are considering quantities with only one variable (e.g., number of boys, number of students, and number of degrees), we use additive thinking, relying on addition and subtraction to answer questions of how many, how much more, and how many fewer. Students start analyzing change between quantities using additive thinking before they start school and with time are able to apply this reasoning to a range of number types and sizes.

However, sometimes we compare the change between quantities in a more complex way, by looking at the multiplicative relationships between quantities. The question *If there are 8 boys in grade 1 and 7 boys in grade 2, are there more boys in the first or the second grade?* calls for comparing the two quantities, 8 and 7. However, if the question instead asked which class has a larger fraction, or percentage, or proportion, of boys, then we'd need to analyze the problem differently and would need to know the total number of children in each class. We might reason: *There are 8 boys out of 20 children in grade 1 ($\frac{8}{20}$, which is less than half) and 7 boys out of 14 children in grade 2 ($\frac{7}{14}$, which is exactly half). So the grade 1 class has a smaller fraction of boys.* Or we might think: *Let's compare 8 to 20, or $\frac{8}{20}$, with 7 to 14, or $\frac{7}{14}$. These ratios are equivalent to 2 to 5 ($\frac{2}{5}$) and 1 to 2 ($\frac{1}{2}$). So in the first grade there are 2 boys out of every 5 students whereas in the second grade there is 1 boy for every 2 students.* Or we could use percents: 40% *of the children in grade 1 are boys and* 50% *of the children in grade 2 are boys.* Using multiplicative thinking, we state that there are more boys in grade 2 because the proportion of boys is greater, but thinking additively there are more boys in grade 1 ($8 - 7 = 1$).

Let's look at another example of how the relationship between two quantities in a ratio conveys different multiplicative information (Lamon 1999):

20 students in a classroom

20 students in the auditorium

20 students in a 10-seat minivan

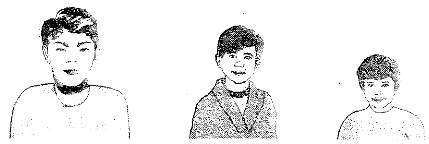
In each of the examples above, the same quantity (20 students) is used. If we just compare the number of students in each situation, we might note that they are the same or that there is no difference between them; we have used additive thinking to compare the quantities. But when 20 students are compared with the other (implied) quantities, each of the comparisons, or ratios, imparts a different meaning that calls for multiplicative reasoning. Twenty students in one classroom is seen by many educators as ideal; an auditorium with 20 students is quite empty compared with the number of seats available; and a 10-seat minivan with 20 students in it is very crowded if not downright unsafe! When we ask which situation is the most crowded, we think multiplicatively and consider both quantities at the same time. Namely, we use multiplication and division in our solution.

Activity

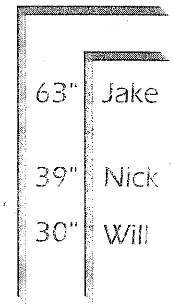
More Than One Way to Compare

Objective: compare situations using both additive and multiplicative reasoning.

Every July 1st, the children in the Hollinger family are measured and the height recorded on the inside sill of their kitchen door. And every July 1st there is a family debate over who grew the most during the past year. Explore this problem in two ways: using additive and multiplicative thinking.



Boys' heights before being measured



Boys' heights after being measured

Things to Think About

Both Jake and Nick grew 3 inches, while Will only grew 2 inches. Using additive reasoning, we can state that two of the boys grew the same amount (and the most that year) and they both grew 1 more inch than Will. But we can consider growth in another way—in relation to their starting heights. When we compare the amount of growth to each boy's starting height, we are using multiplicative thinking. Jake grew $\frac{3}{60}$, or $\frac{1}{20}$, of his starting height; Nick grew $\frac{3}{36}$, or $\frac{1}{12}$, of his starting height; and Will grew $\frac{2}{28}$, or $\frac{1}{14}$, of his starting height. Even though Jake and Nick each grew 3 inches, we can now see that compared with where they started, Nick shot up more relative to his original height. One way to think about this is to compare the simplified ratios: $\frac{1}{20} < \frac{1}{12}$. Using this type of *multiplicative* reasoning, we find that Will also grew relatively more than Jake since he increased his height by $\frac{1}{14}$ compared with $\frac{1}{20}$. Notice that just as we use the operations of addition and subtraction for additive reasoning, we use the operations of multiplication and division for multiplicative reasoning when we compare relative amounts.

Another way to compare the boys' growths multiplicatively is to record them as a percent: $\frac{3}{60} = 5\%$, $\frac{3}{36} = 8\%$, and $\frac{2}{28} = 7\%$. Nick's height increased by 8%, Will's height by 7%, and Jake's height by 5%. If we calculate percents by comparing the new heights with the old heights, we get similar results: $\frac{63}{60} = 105\%$, $\frac{39}{36} = 108\%$, and $\frac{30}{28} = 107\%$. The 107%, for example, indicates that Will exceeded his earlier height (100%) by an additional 7%. Thus, this year the Hollingers agreed that Nick grew the most, followed by Will, followed by Jake—relatively speaking! ▲

Ratios as Rational Numbers

Many ratios belong to the rational number set. Some ratios, however, do not. Rational numbers cannot have zero in the denominator (see Chapter 5, page 100, for more information), but it is possible to have a ratio with zero as the second number; the ratio 11:0 can be used to compare 11 male Boston Red Sox players with 0 female Boston Red Sox players, but it is neither a fraction nor a rational number. Other ratios may involve irrational numbers such as π and $\sqrt{2}$ and also are not part of the rational numbers (see Chapter 1, page 5).

Some interpretations of ratio are closely connected to the meanings of fractions described in Chapter 5 but others are not. For example, we have seen how part-to-whole ratios and fractions are related. Another fraction interpretation, the interpretation of fractions as the result of dividing two numbers, can be connected to ratios. In these instances, ratios are reported as a single number (instead of a comparison) that is created by performing a division. For example, the ratio of circumference to diameter is reported as π and approximated at 3.14. The average number of people per household in the United States was recently reported as 2.57. Batting averages are found by dividing the number of hits by the total number of times at bat (in 1923 Babe Ruth had a batting average of .393). The interpretation of these divided ratios must be considered carefully. Sometimes the divided ratio conveys information as an average (while there aren't 2.57 people in any household, we get a sense of the size of many American households). Other times it describes special relationships about particular geometric figures or about an ability to hit baseballs. Notice that unlike a unit rate where a comparison with 1 is made but not always explicitly stated, some ratios created by performing a division do not convey a comparison of two quantities. Depending on the quantities in the division, some of these divided ratios are rational numbers and some are not.

One aspect of understanding ratios that is related to rational number knowledge involves understanding equivalence. Students benefit from comparing and reasoning about equivalent ratios and rates. The rules for finding equivalent ratios (regardless of the type of ratio) are exactly the same as the rules for finding equivalent fractions. However, equivalent ratios are different in one important way from equivalent fractions. Equivalent fractions, such as $\frac{1}{2}$, $\frac{3}{6}$, and $\frac{40}{80}$, are different symbols that refer to the same quantity or rational number—one half—a number that you can locate on a number line. On the other hand, equivalent part-part ratios do not name the same quantity but rather represent the same comparison between two quantities. The following equivalent ratios—1 girl to 2 boys (1:2), 3 girls to 6 boys (3:6), and 40 girls to 80 boys (40:80)—name a comparison between two quantities, and each also names the same comparison of quantities—there are half as many girls as boys. Likewise, rates such as 125 miles in 5 hours and 50 miles in 2 hours are equivalent because they represent the same hourly speed of 25 mph, but they do not refer to the same distance traveled (125 versus 50 miles). Since some ratios are part-to-whole fractions, we can see how confusing this must be for students. Thus, when discussing the meaning of number sentences, we need to consider the context or situation in which the symbols arise in order to interpret them correctly and consider the type of comparison used (part-whole versus part-part versus rate).

Activity

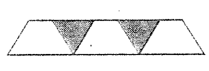
2

Exploring Equivalent Ratios

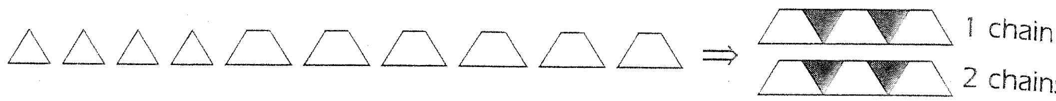
Objective: examine equivalence of ratios.

Materials: pattern blocks.

Using pattern blocks, we can build a polygon "chain" that consists of 2 equilateral triangles and 3 trapezoids, in the ratio of 2:3:



If we have 4 triangles and 6 trapezoids, we can build two chains:



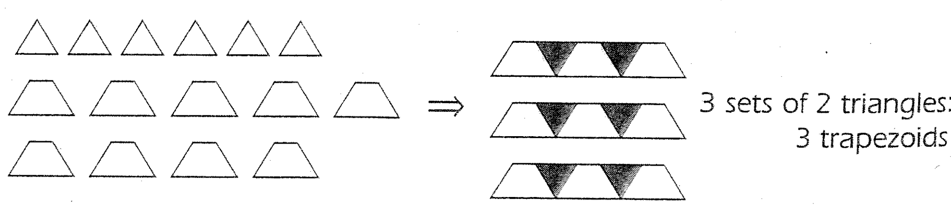
Using the ratio of 2 triangles for every 3 trapezoids, complete the table below and answer the questions that follow.

NUMBER OF CHAINS	NUMBER OF TRIANGLES	NUMBER OF TRAPEZOIDS	RATIO OF TRIANGLES TO TRAPEZOIDS
1	2	3	2:3
2			
3			
4			
		18	
	135		
			32:48

1. If the number of trapezoids used to make chains is 18, how many triangles are needed? What is the ratio of triangles to trapezoids? Is this ratio equivalent to 2:3? Why or why not? Is this ratio a part-whole or part-part ratio?
2. If you want to use 135 triangles, how many trapezoids will you need so the relationship between triangles and trapezoids is maintained?
3. If the total number of blocks is 65, how many triangles and trapezoids are there?

Things to Think About

Did you notice any patterns in the table? The numbers of triangles are multiples of two, and the numbers of trapezoids are multiples of three. Two chains have a triangle/trapezoid ratio of 4:6, and three chains have a ratio of 6:9. In addition, all the ratios of triangles to trapezoids (2:3, 4:6, 6:9, etc.) are equivalent. Why? Because each subsequent ratio compares a different number of triangles and trapezoids, but all the comparisons are multiples of the original comparison, the ratio 2:3. Put differently, we can take 6 triangles and 9 trapezoids and arrange them to show the same comparison of quantities—3 chains of 2 triangles and 3 trapezoids.



If 18 trapezoids are used to make polygon chains, there are six 2-triangle-3-trapezoid chains. Thus, six sets of 2 triangles, or 12 triangles, are needed, and the ratio can be written as 12:18. Tables like the one above are helpful tools for organizing one's work, but may not lead to developing students' understanding of the multiplicative relationships inherent in proportional situations. This is especially true if students successively add 2 in the triangle column and 3 in the trapezoid column, building up the values in each column until they reach 12 triangles and 18 trapezoids: there is some question whether students see multiplicative relationships in situations that they solved using addition strategies. One way to promote multiplicative reasoning is to introduce students to ratio tables that display equivalent ratios, with the guideline that predominately multiplication and division should be used to find equivalent values. The focus is on the relationship between the two quantities in the ratio and on operating on both of these quantities in order to form equivalent relationships that have the same comparison of quantities. Two different ratio tables that might be used to solve this problem are shown below.

RATIO TABLE A

	$\times 3$	$\times 2$	
TRIANGLES	2	6	12
TRAPEZOIDS	3	9	18
	$\times 3$	$\times 2$	

RATIO TABLE B

	$\times 6$	
TRIANGLES	2	12
TRAPEZOIDS	3	18
	$\times 6$	

How many trapezoids are needed if there are 135 triangles? Did you divide 135 by 2 (67.5) and then multiply that quantity by 3? If we take 67.5 sets of the 2:3 ratio, we end up with an equivalent ratio of 135:202.5. If 135 triangles are used to make chains, we can make 67 complete chains and half of another chain! One way students might approach this problem is to analyze the quantities and then use a ratio table. There are many correct ways to build a ratio table, and as can be seen, to combine operations to solve problems. In Table A, a student first multiplied 2:3 by 135 to get an equivalent ratio of 270:405, then subsequently divided each part of the ratio by 2 to end up with 135 compared with 202.5. The student using Table B multiplied the 2:3 ratio twice and then combined the parts. First, she multiplied 2:3 by 60 to get the equivalent ratio 120:180. Then, instead of continuing from 120:180, she again multiplied the ratio 2:3 by $7\frac{1}{2}$ to produce the equivalent ratio $15:22\frac{1}{2}$. By combining the ratios, 120:180 and $15:22\frac{1}{2}$, she ended up with 135 to $202\frac{1}{2}$.

RATIO TABLE A

	$\times 135$	$\div 2$	
TRIANGLES	2	270	135
TRAPEZOIDS	3	405	202.5
	$\times 135$	$\div 2$	

$\left(135 \div 2 = 67\frac{1}{2}\right)$

$\frac{2}{3} \times 67\frac{1}{2}$

RATIO TABLE B

	$\times 7\frac{1}{2}$			
	$\times 60$	+		
TRIANGLES	2	120	15	135
TRAPEZOIDS	3	180	$22\frac{1}{2}$	$202\frac{1}{2}$
		+		
	$\times 60$	$\times 7\frac{1}{2}$		

$\left(\frac{2}{3} \times 60\right) + \left(\frac{2}{3} \times 7\frac{1}{2}\right) \Rightarrow 135:202\frac{1}{2}$

$\frac{2}{3} \times 67\frac{1}{2}$

If the total number of blocks is 65, how many triangles and trapezoids are there? One way to answer this question is to extend the table and record the number of triangles and trapezoids until a total of 65 blocks are used. Another way involves noting that 5 blocks are needed for the ratio 2:3 and there are 13 sets of 5 in 65. Since in the original ratio there are two triangles to three trapezoids and there are 13 sets of this ratio, you can multiply each of these parts by 13 (2 × 13 and 3 × 13) to find the equivalent ratio, 26:39. Another approach is to use a ratio table with three rows (triangles, trapezoids, total blocks). Try using the ratio table as a tool for solving this problem.

TRIANGLES	2	20	6	26
TRAPEZOIDS	3	30	9	39
TOTAL BLOCKS	5	50	15	65

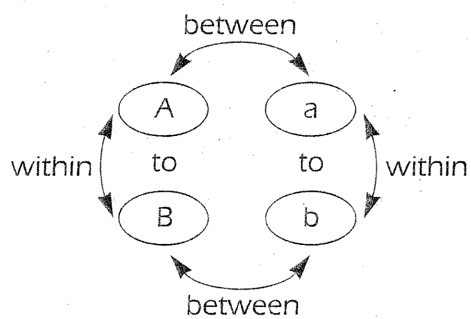
If there are 65 blocks, 26 are triangles and 39 are trapezoids. ▲

We do not add and subtract part-part ratios in the same way we add and subtract fractions. When adding or subtracting fractions, we are interpreting the quantities as parts of wholes and are joining parts to make wholes. The triangle/trapezoid ratio in Activity 2 is not a comparison of parts with a whole but a comparison between a part and another part. When these part-part ratios are combined, the end result is another equivalent comparison, not a total. When we combine two polygon chains (2:3 + 2:3), we write the resulting ratio of triangles to trapezoids as 4:6, which means four triangles compared with six trapezoids; when we combine two fractions $(\frac{2}{3} + \frac{2}{3})$, we are finding the total amount. The sum is $\frac{4}{3}$, or $1\frac{1}{3}$, which means one whole unit and one third of a second unit. Therefore, contextual information is very important and aids in helping students focus on the type of relationship they are considering—a comparison between parts and parts rather than between parts and a whole.

Proportions

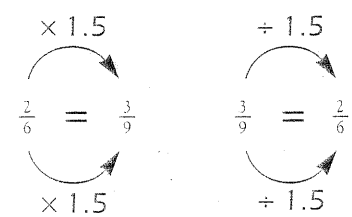
Lessons about proportions are often included in instructional materials about ratios and rates in the middle grades. A *proportion* is a mathematical statement of equality between two ratios. Stated another way, proportions tell us about the equivalence of ratios. For example, $\frac{6}{9} = \frac{8}{12}$ is a proportion. Both ratios can be simplified, indicating they represent the same comparison: $\frac{2}{3} = \frac{2}{3}$. The fractional form of writing proportions is generally preferred since it is more suitable for solving equations. However, proportions can also be presented using ratio notation: 6:9 = 8:12 or 6:9 :: 8:12. The double colon indicates the ratios are equivalent. When proportions are presented devoid of context, the symbols can be interpreted as equivalent fractions or as equivalent ratios (which includes rates).

A proportion includes multiplicative relationships "within" the individual ratios and multiplicative relationships "between" the two ratios. For example, to make an apple pie we need 2 cups of sugar for every 6 cups of apples, or 3 cups of sugar for every 9 cups of apples. The proportion $\frac{2}{6} = \frac{3}{9}$ represents this relationship. The "within" ratios are 2:6 and 3:9 and both simplify to 1:3, or $0.\overline{3}$. The "between" ratios are 2:3 and 6:9 and both simplify to 2:3, or $0.\overline{6}$. The diagram below illustrates that in a proportion, the two "within" ratios are the same and the two "between" ratios are the same.



A:B and a:b within ratios
A:a and B:b between ratios

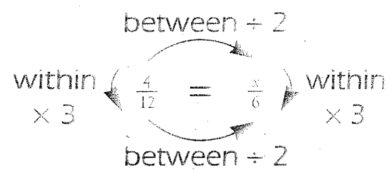
There is another "within" ratio relationship: if we multiply the first number in the 2:6 ratio by 3 we will obtain the second number in the ratio ($2 \times 3 = 6$). If we examine the multiplicative relationship "between" the ratios, we discover that multiplying both numbers in the first ratio by 1.5 ($2 \times 1.5 = 3$; $6 \times 1.5 = 9$) gives us the second ratio. Likewise, dividing both numbers in the second ratio by 1.5 returns us to the first ratio.



This quantity (1.5) is known as a *scale factor* and is discussed in detail on page 178. Choose a different scale factor, such as 2, and make another proportion ($\frac{2}{6} = \frac{4}{12}$). Can you figure out why these two ratios are equivalent?

A characteristic of proportions is that the "cross products" are equal. Using the proportion $\frac{2}{6} = \frac{3}{9}$, this means that 2×9 should equal 6×3, which is true: $18 = 18$. Let's explore why this occurs. One way to think of it is that since the ratios are equivalent, they simplify to an identical ratio—the "within" ratio—an identical comparison of quantities—in this case 1 to 3—and $\frac{1}{3} = \frac{1}{3}$. Likewise, both of the ratios can be rewritten as a comparison of 6 to 18 ($\frac{6}{18}$). This equivalent ratio is found by taking three sets of the ratio 2:6 and two sets of the ratio 3:9. The first number of one ratio indicates how many sets of the other ratio are needed to make them equivalent. As with $\frac{1}{3} = \frac{1}{3}$, the cross products for $\frac{6}{18} = \frac{6}{18}$ are the same.

Unfortunately, students are taught how to use cross products to find missing values ($\frac{4}{12} = \frac{x}{6}$, $12x = 24$, $x = 2$) before they have developed an understanding of proportional situations. As a result they apply the cross product algorithm to any and all situations even though cross products are only equal when the ratios are equivalent—the relationship doesn't hold for all pairs of ratios! Students instead should first have extensive opportunities to solve problems that involve proportions and multiplicative thinking using their own ideas and methods before being introduced to the cross product algorithm. For example, the value of x can be found in the proportion above by using multiplicative relationships in the "within" ratios or the "between" ratios.



The phrase *proportional reasoning* is used when describing the thinking that has been applied to the solution of problems that involve multiplicative relationships. Topics studied in upper elementary and middle school such as fractions, percents, ratios, similarity, scale indirect measurement, and probability involve some aspects of proportional thinking. That said, proportional reasoning involves more than simply using a proportion or cross products as part of a solution. An important step in understanding proportions occurs when students think of a ratio as an entity unto itself—they don't just consider the two quantities that make up the ratio separately but are focused on the relationship between the quantities. In addition, when students start to recognize proportional situations in everyday settings and to reason multiplicatively about them, they are developing insight into this topic. Teachers should provide students with a variety of problems to solve in these areas starting in about fourth grade, because it is now known that it takes a number of years for students to develop the ability to use and thus be able to reason proportionally. One major recommendation from many researchers is that we should hold off teaching the standard algorithms for operating on fractions and solving proportions until students have made sense of some of the fundamental concepts.

2. Applications of Ratios and Rates

Proportional reasoning is an essential component of many courses in mathematics and science, including algebra, geometry, chemistry, and physics. Ratios and rates are used to answer questions about unit pricing, population density, percents (see Chapter 7), speed, slopes, conversion of money, map reading, reductions and enlargements of figures, fractions (see Chapter 5), and drug concentrations.

Similarity

In everyday life we talk about things being "similar" or "a little bit alike," but the mathematical meaning of this term refers to a particular geometric concept. What does it mean for two shapes to be similar? Students in grade 5 or 6 are often introduced

3 Exploring Similar Figures on a Coordinate Grid

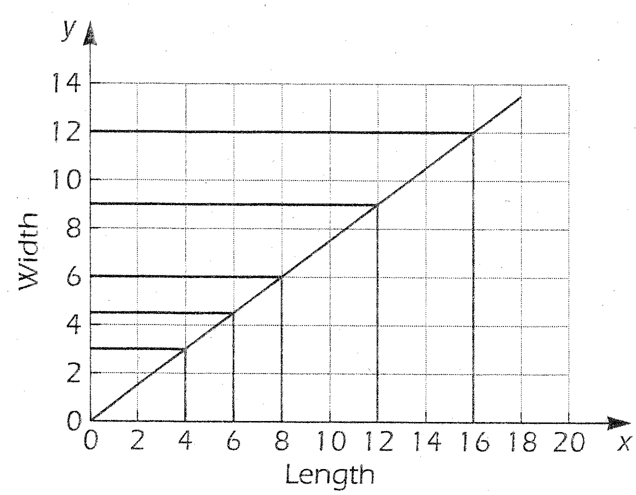
Objective: explore how equivalent ratios can be expressed arithmetically, geometrically, and algebraically.

Materials: graph paper.

Draw a graph of the first quadrant and label the x-axis "length" and the y-axis "width." Draw the rectangles with the following dimensions (width to length) on the coordinate grid: 3×4, 6×8, 8×10, 9×12, and 12×16. Each rectangle should start at the origin (0,0) and its shorter side should align with the y-axis and its longer side should align with the x-axis. Next draw a line from the origin through the upper right corner of the 12-by-16 rectangle. Which rectangles are similar to one another? Which width-to-length ratios are equivalent? How are the width-to-length ratios shown on the coordinate grid? Draw another line from the origin through the upper right vertex (corner) of the 8×10 rectangle and use it to determine the dimensions of four rectangles similar to it.

Things to Think About

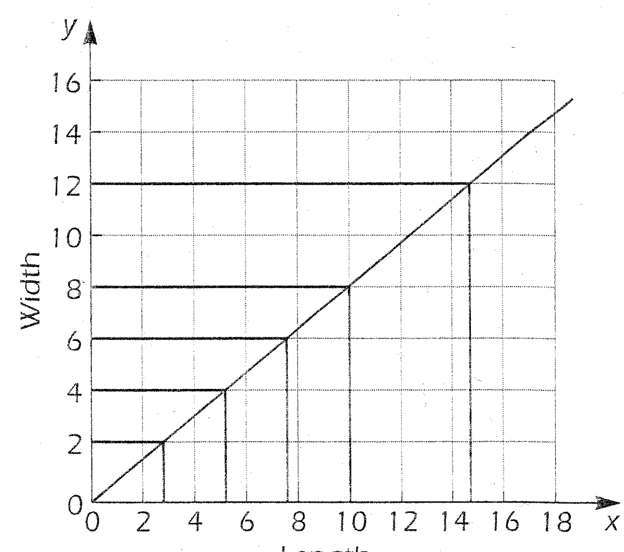
Did your line go through the upper right corner of the 3×4, 6×8, 9×12, and 12×16 rectangles? This is because the width-to-length ratios of these rectangles are equivalent—they name the same comparison between quantities, since each of these ratios can be simplified to the "within" ratio of .75:1. In addition, we can say that these four rectangles are all similar because they have corresponding angles that are congruent and corresponding dimensions that are proportional. Using this line we can find other rectangles that are similar to these four. Take a point on the line such as (6, 4.5) and draw vertical and horizontal lines to the axes. The resulting rectangle, 4.5×6, is similar to the other rectangles. These multiplicative relationships can be expressed arithmetically using width-to-length equivalent ratios but also geometrically using similar rectangles nested within each other.



The slope of this line is $\frac{3}{4}$, or .75. The slope, which is a comparison of vertical change with horizontal change, algebraically connects these equivalent ratios

Ratios that are equivalent, such as 3:4, 4.5:6, 6:8, 9:12, 12:16, and many more, all fall on the same line on the grid.

The width-to-length ratio of the other rectangle, 8 × 10, does not name the same comparison. Let's find other ratios (and thus rectangles) that are equivalent (or similar) to 8:10. One way is to use a ratio table to list equivalent ratios such as 4:5 or 12:15. Another is to draw a line from the origin (0,0) through the point (10,8) and extend it. The coordinates of points on the line represent width-to-length ratios equivalent to 8:10. If you draw vertical and horizontal lines from points along the line, you create similar rectangles, such as 2 × 2.5, 4 × 5, 6 × 7.5, and 12 × 15. The slope of this new line is $\frac{4}{5}$—the simplified "within" ratio for this set of equivalent ratios.



Rectangles similar to 8:10
(width:length)

Draw a line from the origin with slope $\frac{3}{4}$ on this second graph. How do the different lines compare? Notice that the line with slope $\frac{4}{5}$ is steeper. Can you estimate where a line representing all equivalent ratios to 9:10 would go on the graph? ▲

A common application of the principles of similarity involves making scale drawings and models. Scale drawings are drawings of similar figures that have either been enlarged or reduced. Drawings of microscopic organisms are enlarged in biology books, while pictures of our solar system are reduced. Three-dimensional scale models are used in automobile and aircraft design and in architectural plans. Students may have read about both enlargements and reductions in children's literature (*Gulliver's Travels*, *The Littles*) and may have firsthand experience with models in the form of miniature cars and train sets. Some of your students may have tried to make sense of a set of blueprints or to enlarge a drawing in art class.

How do we make enlargements and reductions? To create a scale drawing, we need a *scale factor*. The scale factor is a number by which all of the dimensions of an original figure are multiplied to produce the dimensions of the enlargement or reduction. When the scale factor is greater than 1, the new figure, sometimes referred to as an *image*, is an enlargement. When the scale factor is a number between 0 and 1, the resulting figure will be reduced in size, and is often referred to as a *reduction*. Scale factors use the operator interpretation of rational number in which the fraction acts either to stretch (enlarge) or to shrink (reduce) all dimensions of a drawing or three-dimensional model (see Chapter 5, page 108, for additional information about this interpretation). Mathematicians often use the letter k to stand for the scale factor.

How do the perimeters and areas of similar figures compare? What relationship exists between the volumes of enlarged or reduced figures? These questions are explored in the following activities.

Activity 4

Perimeters and Areas of Squares

Objective: look for patterns in the perimeters and areas of similar squares.

Materials: graph paper.

What happens to the perimeter of a square when the dimensions (sides) of that square are doubled? tripled? quadrupled? What happens to the area of a square when its dimensions (sides) also are doubled? tripled? quadrupled? Investigate doubling dimensions by drawing a square of any size and calculating its perimeter and area. Next double the length of the sides of that square and calculate its "doubled" perimeter and area. Do this for several squares, keeping track of your measurements and looking for patterns in the "doubled" data. Then investigate what happens to the perimeter and area of squares if you triple or quadruple the side lengths.

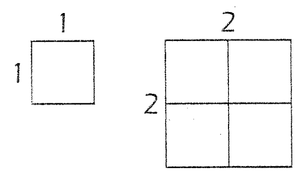
Things to Think About

Did you notice that by doubling and tripling dimensions, you were making enlargements? What is the scale factor, k, in each case? When the sides are doubled, $k = 2$; tripling dimensions means that $k = 3$; and quadrupling gives us $k = 4$. Did you notice that when the side lengths were doubled, the perimeter was 2 times longer? When the side lengths were tripled the perimeter was 3 times longer, and when the sides were quadrupled, the perimeter was 4 times longer than the original perimeter. We can generalize that the perimeter of an enlarged square with scale factor k is equal to k times the original perimeter. Take a minute and explain why.

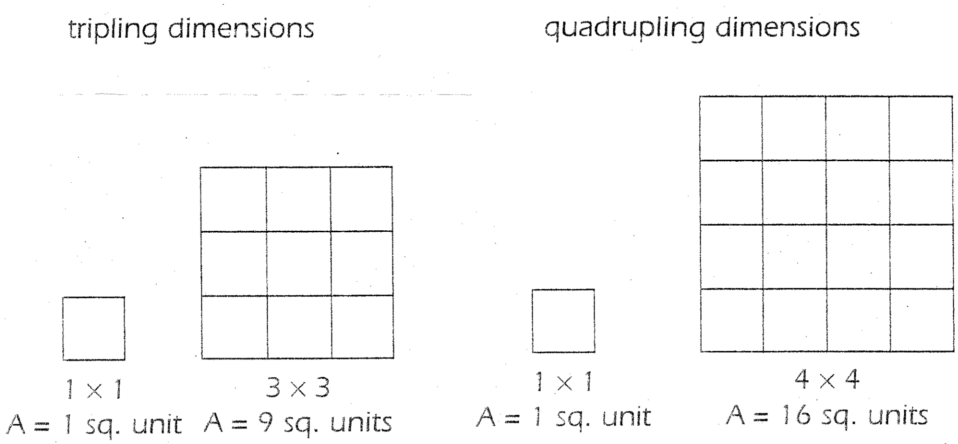
When the dimensions or sides of a square are doubled, the new area is 4 times as large; when the side lengths are tripled, the new area is 9 times as large; and when the side lengths are quadrupled, the new area is 16 times as large. Why? Let's use a 1-by-1 square to explain. Doubling each dimension means that the new dimensions are 2 by 2, or (1×2) by (1×2). The area of this "doubled" square can be found through this series of calculations:

Original Area	$= 1 \times 1$	
Doubled Area	$= 2 \times 2$	
	$= (1 \times 2) \times (1 \times 2)$	double each side of the original square
	$= (1 \times 1) \times (2 \times 2)$	rearrange the order and grouping of the factors
	$= 1 \times 4$	original area times 4
	$= 4$	the new area is 4 times greater than the original area

This can also be shown by analyzing the following diagram:



Notice that four 1-by-1 squares fit in the "doubled" square. Since each dimension is doubled ($k = 2$), the resulting area is quadrupled (2×2 or $k = 4$). The diagrams below show why the area of a square is 9 times greater when each side is tripled and 16 times greater when each side is quadrupled.



Activity 5

Areas of Similar Figures

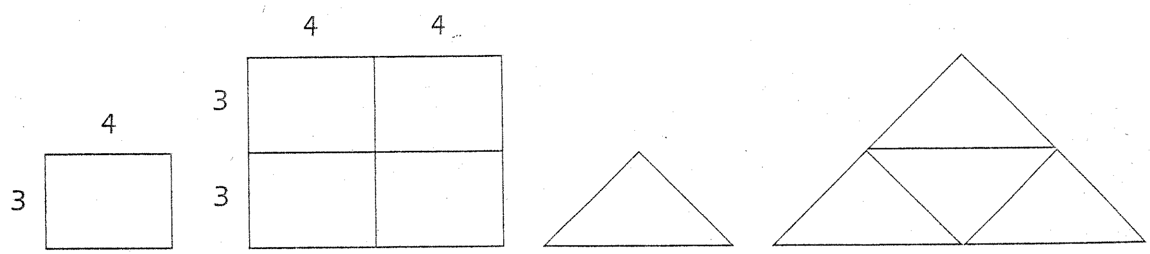
Objective: generalize how the scale factor affects the area of similar figures.

Do the relationships between areas that result when the dimensions of squares are doubled, tripled, and quadrupled hold true for rectangles, triangles, and circles? That is, do the areas of other similar figures using scale factors of 2, 3, and 4 follow what we observed with squares? Using a calculator to find areas (area of rectangle = lw; area of triangle = $\frac{1}{2}bh$; area of circle = πr^2), complete the "doubling" table on page 180. Each time you double, triple, or quadruple all of the dimensions, the angles in the enlarged figures should be congruent (have exactly the same measure) with the angles in the original shapes, as you are creating similar figures. Examine the data for patterns. Construct your own tables for "tripling" and "quadrupling" dimensions. Explain the patterns you observe.

SHAPE	DIMENSIONS (IN CM)	AREA (IN CM^2)	DIMENSIONS DOUBLED (IN CM)	DIMENSIONS DOUBLED AREA (IN CM^2)	RATIO OF DIMENSIONS DOUBLED AREA TO AREA
Rectangle	$l = 4, w = 3$		$l = 8, w = 6$		
Rectangle					
Triangle	$b = 5, h = 4$		$b = 10, h = 8$		
Triangle					
Circle	$r = 3$		$r = 6$		
Circle					

Things to Think About

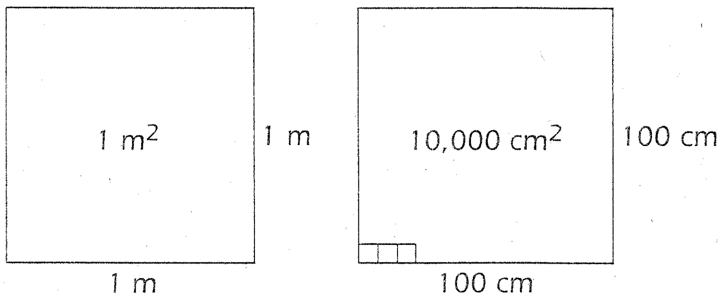
The relationships you observed with squares does hold true for rectangles, triangles, and circles. Doubling the dimensions of these shapes results in areas that are 4 times as large as the original areas. Likewise, tripling and quadrupling dimensions leads to areas of the enlarged shapes being 9 and 16 times larger, respectively. Stated in another way, if the scale factor is k, the area of the enlarged figure is k^2 times the original area. Does this relationship hold for scale factors that are less than 1? Yes. Let's say the scale factor is $\frac{1}{2}$. This means we multiply both dimensions by $\frac{1}{2}$ for a combined reduction of $(\frac{1}{2})^2$, or $\frac{1}{4}$. In summary, the area of a scaled figure is k^2 times the original area, where k is the scale factor, regardless of the size of k or the original area of the figure. A diagram might help you visualize these relationships.



A circle doesn't have two dimensions like those of length and width. Why then is the area of the circle four times greater when the radius is doubled? In the formula for finding the area of a circle ($A = \pi r^2$), notice that the radius is squared. This means that you multiply the radius by itself, rather than having two different dimensions such as length and width. When you double and then square the radius, the new area is $(2r)^2$, or $4r^2$. The relationship holds: the area of the scaled circle, using scale factor k, is k^2 times the original area. ▲

The relationship between the dimensions of a figure and the area of that figure is very similar to the relationship that occurs when one square unit of measurement is converted to another. One square meter is equivalent in size to a square that is

100 centimeters per side. Since a smaller unit of measurement is used to determine the length of each side, the numerical value of the area based on these smaller square units will be greater—10,000 (100 × 100) times greater! The area of the original square meter doesn't change, but the size of the unit used to measure the area does.



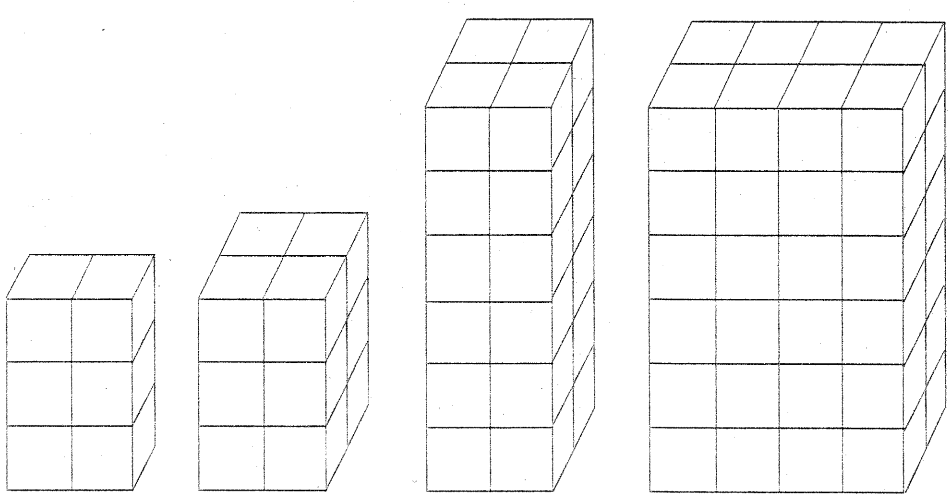
Activity

Doubling Dimensions of a Solid

Objective: explore the relationship between volumes and scale factors of similar figures.

Materials: multilink or unit cubes.

What happens to the volume of a rectangular prism if one, two, or all dimensions are doubled?



$2 \times 3 \times 1$ $V = 6u^3$	$2 \times 3 \times \underline{2}$ double one dimension $V = 12u^3$	$2 \times \underline{6} \times \underline{2}$ double two dimensions $V = 24u^3$	$\underline{4} \times \underline{6} \times \underline{2}$ double three dimensions $V = 48u^3$

You can use unit cubes, drawings and sketches, or numerical data to explore the changes as you double one dimension at a time. You may find it easier to recognize relationships and patterns if you organize your data in a table like the one on page 182. In the example given, each doubled dimension is underlined. You may also wish to use unit cubes or multilink cubes to build one prism and the subsequent "doubled" prisms in order to "see" the results of doubling dimensions.

DIMENSIONS OF PRISM	ORIGINAL VOLUME (CUBIC UNITS: U^3)	VOLUME AFTER ONE DIMENSION DOUBLED (CUBIC UNITS: U^3)	VOLUME AFTER TWO DIMENSIONS DOUBLED (CUBIC UNITS: U^3)	VOLUME AFTER THREE DIMENSIONS DOUBLED (CUBIC UNITS: U^3)
$1 \times 2 \times 5$	10	$2 \times 2 \times 5 = 20$	$2 \times 4 \times 5 = 40$	$2 \times 4 \times 10 = 80$

If you triple all the dimensions of the prism, how many times greater is the volume of the prism? How are the volumes of similar figures related?

Things to Think About

What happens to the volume when one dimension is doubled? two dimensions are doubled? all three dimensions are doubled? The volume changes by powers of two. Doubling one dimension doubles the volume ($\times 2$); doubling two dimensions quadruples the volume ($\times 2 \times 2$); doubling all three dimensions results in an eightfold increase in volume ($\times 2 \times 2 \times 2$). This final volume ($\times 8$) is the result of each dimension (length, width, and height) being multiplied by two (doubled), and then multiplied together.

When all three dimensions are tripled, the volume becomes $3 \times 3 \times 3$, or 27 times greater, since each dimension is multiplied by 3. The volumes of similar figures are related by the scale factor. If the scale factor is 2, the new volume is 2^3, or 8 times greater. If the scale factor is 3, the new volume is 3^3, or 27 times greater. We can say that when the scale factor is k, the volume of the similar figure is k^3 times the original volume. The relationship holds for reductions as well. If a scale factor is $\frac{1}{4}$ then the volume of a reduced figure is $(\frac{1}{4})^3$—or $\frac{1}{64}$—times the original volume.

The relationship between increasing dimensions and volume can be used to explain why the existence of giants is mathematically questionable. Imagine a six-foot, 200-pound man and consider how he is similar to a rectangular prism. He might measure two feet across the chest, one foot front to back, and we know he is six feet tall. If we double all three of those dimensions, the resulting giant is four feet across at the chest, two feet front to back, and twelve feet tall. The volume of the giant, however, is eight times greater. Because of the relationship between volume and mass we can state that the mass of the giant is 1600 pounds. This weight is too heavy for human bones (even big ones) to support! Using this same line of reasoning, it also is unlikely that a giant-sized grasshopper would be possible. In this case, the large size and weight of a giant grasshopper couldn't be sustained by its delicate structure and lack of an internal sketeton. ▲

Unit Rates

When a rate is simplified so that a quantity is compared with 1, it is called a *unit rate*. Unit rates answer the question *how many (or how much) for 1?* Some unit rates are constant. This means that the simplified rate does not change (e.g., there are 2.54 centimeters for every 1 inch). Conversions between measurements (inches to feet, ounces to pounds, kilograms to pounds) are examples of constant unit rates. Other unit rates vary. The most common example of a varying rate is the monetary exchange rate. The rate of 1.20 Euros for 1 U.S. dollar is not fixed. Six months from now the rate may be 1.34 Euros for 1 U.S. dollar or it may be 1.05 Euros for 1 U.S. dollar! Whether or not a rate is constant or varies is rarely addressed in instructional materials, but it is an important topic for students to discuss.

Sometimes a rate is expressed as a single number. As seen earlier, single number rates are created by dividing one quantity by another. Some of these single number rates are actually unit rates, where the unit is implied but not explicitly stated. In other cases the comparison is not so clear. For example, in the paper we might learn about the death rate in a particular country being 6.8. Death rates are comparisons to 1000 people so this is not a unit rate (nor is it actually a rate, since people are being compared to people!). On the other hand, a heart rate of 130 is a unit rate, because the comparison is the number of heart beats with 1 minute of elapsed time. In general, it is difficult to establish whether or not a single number is a unit rate without investigating how the rate was derived. Unemployment rates, postal rates, mortgage rates, and inflation rates are interesting to research and discuss.

One of the interesting things about unit rates is that there are two ways to express the relationship. For example, at a local farm stand, tomatoes are $1.49 per pound. Most of us are very familiar with this type of unit rate from grocery shopping, where the unit refers to the number of pounds (1 pound of tomatoes for $1.49). But what if the unit refers to the number of dollars? How many pounds of tomatoes can you buy for $1.00? We can buy about 0.67 pound of tomatoes for $1.00. Usually, students find one form of a unit rate easier to interpret, but the context of when and how the rate is being used makes a difference. In this next activity, think about which unit rates make the most sense to you and why.

Activity 7

Unit Rates

Objective: understand the dual nature of unit rates.

Express the following ratios as two different unit rates. Try using a variety of tools to help you determine the unit rate, such as ratio tables, pictures, or graphs. Which form of each ratio makes the most sense? Why?

120 miles in 2 hours
5 pizzas for 3 teenagers
$42 for 7 watermelons
50 GBP (British pounds) being equivalent to $88.48 USD (U.S. dollars)
20 candies for $2.50

Things to Think About

Two different unit rates are possible for each of the above ratios:

60 miles in 1 hour	or	1 mile in $\frac{1}{60}$ hour (1 minute)
$1\frac{2}{3}$ pizzas for 1 teenager	or	1 pizza for $\frac{3}{5}$ teenager
$6 for 1 watermelon	or	$1 buys $\frac{1}{6}$ watermelon
1 GBP is equivalent to $1.7696 USD	or	$1 USD is equivalent to 0.5651 GBP
1 candy for $12\frac{1}{2}$¢	or	$1 buys 8 candies

Some unit rates make no sense, such as 1 pizza for $\frac{3}{5}$ teenager. And some unit rates are more practical in one form than in another, such as miles traveled in 1 hour. However, often both unit rates provide useful information. In a candy store, we sometimes may want to spend \$1.00; at other times, when we only want a few pieces, it is useful to know the cost of 1 candy. When visiting another country we need to be able to convert currencies depending on the circumstance (*I just arrived in London. How many GBP will I get for \$25?* or *I am about to get on the plane for home and I didn't spend 6 GBP. How many U.S. dollars have I tied up in GBPs?*).

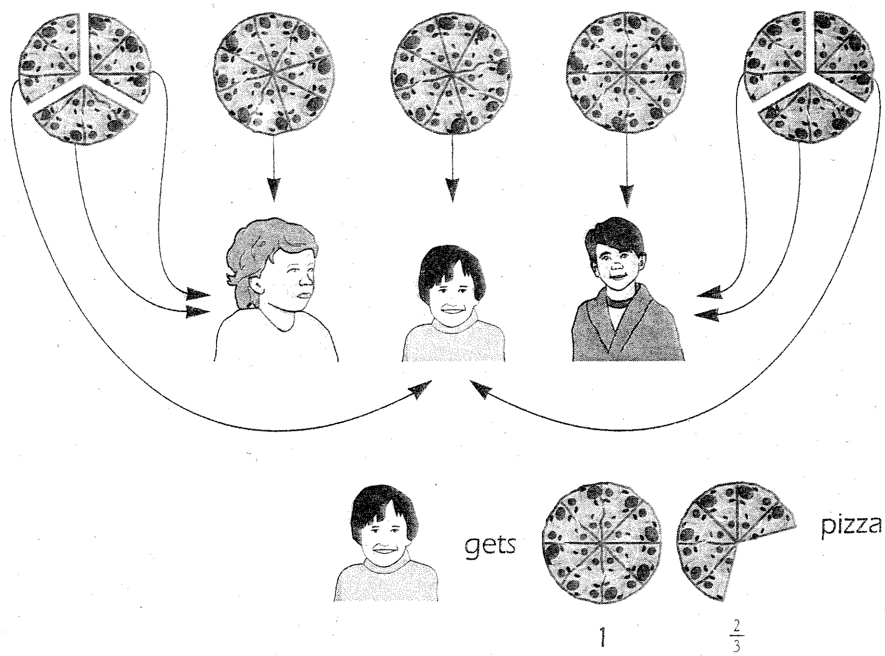
How did you find the unit rates? Constructing a ratio table and then dividing is one approach. Labels are essential in order to keep track of the relationships. For example:

		÷ 7	÷ 6
DOLLARS	\$42	\$6	1
WATERMELONS	7	1	$\frac{1}{6}$
		÷ 7	÷ 6

It helps if you decide ahead of time which variable you wish to be 1 and then think about the operation that will produce that result.

Students often use pictures to make sense of these relationships, especially when the numbers are small and easy to manipulate.

5 pizzas for 3 children



Each child gets $1\frac{2}{3}$ pizza. ▲

Sometimes finding a unit rate is an especially efficient method for solving a proportion problem. For example, if 4 gallons of gasoline cost \$10 and we need to calculate the cost to fill up a 17-gallon tank, one method is to find the unit cost (\$2.50 per gallon) and multiply that amount by 17 (\$2.50 × 17 = \$42.50). However, unit rates should not be used to the exclusion of other methods. A similar problem—if 4 gallons of gasoline cost \$10, how much will 16 gallons cost?—can be solved quite differently and more easily. Notice that 4 groups of the rate 4 to 10 is equivalent to 16 gallons for \$40. By multiplying the initial rate by a factor of 4 ($\frac{4 \text{ gallons}}{\$10} \times \frac{4}{4} = \frac{16 \text{ gallons}}{\$40}$), we have saved time and mental energy and perhaps avoided error since 10 × 4 is so straightforward! The second method is sometimes referred to as a *factor-of-change method*. The factor-of-change method could have been used to solve the first problem ($\frac{4 \text{ gallons}}{\$10} \times \frac{4.25}{4.25} = \frac{17 \text{ gallons}}{\$42.50}$), but the numbers in that problem don't lend themselves to that method; we are not likely to recognize that 4.25 is the correct factor of change, nor multiply by that factor mentally. Try to give students problems with numbers that lend themselves to the use of both approaches (not in the same problem) and encourage discussions about how we should choose a solution method based on the numbers in a problem.

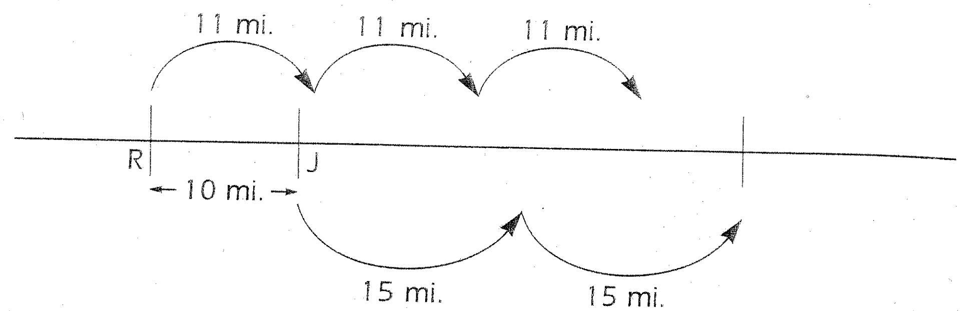
Distance, Rate, and Time

The relationships among distance, rate, and time are known by the formula $D = rt$. Students learn about distance from personal experience getting to different locations on foot, by bicycle, or in a car. They realize that it takes different amounts of time, depending on the mode of transportation, to cover the same distance. Many students do not understand that *speed* is the common term for rate and is actually the comparison of distance with time. Since speedometers give us speed as one number, we tend not to think of it as miles per hour. In terms of instruction, researchers suggest that students in grades 6 through 8 would benefit from thinking about and discussing ways of comparing speeds, the difference between constant speed and average speed, and how rates are used to measure speed. Lamon (1999, 215) states: "Knowing the rule [$D = rt$] does not provide the level of comprehension needed to solve problems. We want students to develop an understanding of the structure of this set of relationships that comes from, but goes beyond, the investigation of specific situations. . . . [T]hey should be able to make generalizations such as this: if distance doubles, time will have to double if speed remains the same, or speed has to double if time remains the same."

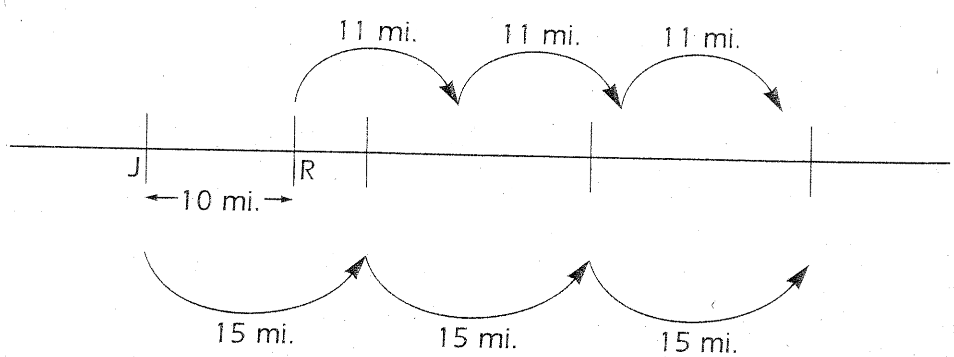
One way to expand students' interpretation of the distance-rate-time relationship is to use graphs to compare different speeds. Two bicyclists, who live 10 miles apart on the same road, follow the exact same route every Sunday morning. Roberta's average speed on this route is 11 mph and Jeff's average speed is 15 mph. They both leave at 8:00 A.M. They meet at the same location every Sunday for a cup of coffee. How far have they biked and what time is it when they meet for coffee?

There are a lot of things to consider in this problem. First, what does it mean that their average speeds are 11 and 15 miles per hour, respectively? Average speed implies that if we could have maintained a constant speed for the whole distance, this would be it! When biking we slow down, speed up, stop, and sometimes maintain a constant speed. Average speed is the total distance traveled compared with the total time it takes us to travel that distance (rather than the average of all of the individual segments of a trip). Since Roberta and Jeff bike this route regularly, they know their average speeds. When we graph average speed, we treat it as if the speed were constant but it is important to realize that it really is not.

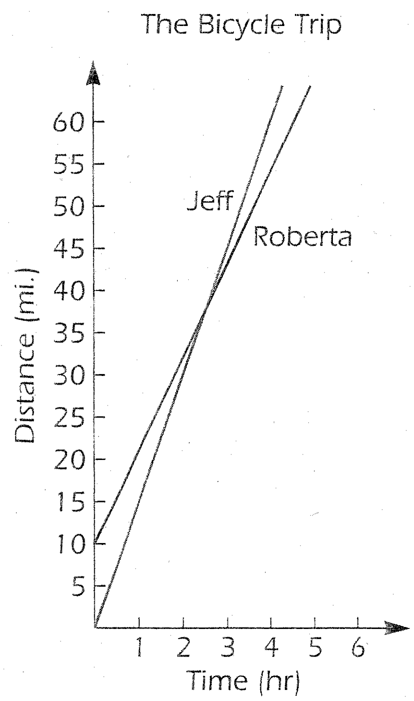
Second, when graphing distance-rate-time relationships, which variable goes on each axis? Usually time is on the x-axis as it is the independent variable (see Chapter 9). Distance traveled depends on the time and is on the y-axis. To graph these speeds we plot (time, distance) points such as (1, 15), which represents that after 1 hour, Jeff has traveled a total of 15 miles. Did you notice that Roberta and Jeff live 10 miles apart? Who lives farther away from their coffee spot? It is not explicitly defined in the problem, although we are told that they always meet there. Since Jeff's speed is faster, if Roberta lives farther away, they could never arrive at the same time.



However, if Jeff lives farther away, then he might be able to catch up to her since he bicycles at a faster speed.



How is this difference in starting location shown on a coordinate graph? The graph of Jeff's trip starts at (0,0), meaning he has traveled 0 minutes and gone 0 distance at the beginning of the bike ride. We start the graph of Roberta's trip at (0,10); she has traveled 0 minutes but is already 10 miles closer to the coffee shop since they are starting at different locations along the route. The line representing Jeff's trip increases by 15 miles for every hour while the line representing Roberta's trip increases by 11 miles per hour. Why are these trips shown as straight lines? Remember we are interpreting average speed as constant speed even though we know in reality that this isn't the case. The graph of the two trips is shown below.



Notice that the line representing Jeff's speed is steeper than the line representing Roberta's speed. What does this mean in reality? Does Jeff go farther? Does he ride faster? Is he riding uphill? Students sometimes interpret graphs as pictures rather than as relationships between variables (students think that the steepness of the line representing Jeff's speed indicates he is going up a steep hill). Answering these questions, examining the relationship between distance and time, and linking the responses to the slopes of the lines ($\frac{distance}{time}$, or $\frac{15}{1}$ and $\frac{11}{1}$) aids in developing students' understanding of speed as a comparison of distance with time. When do Roberta and Jeff meet for coffee? Two and one-half hours after starting, when Jeff will have ridden 37.5 miles and Roberta will have ridden 27.5 miles.

Activity 8

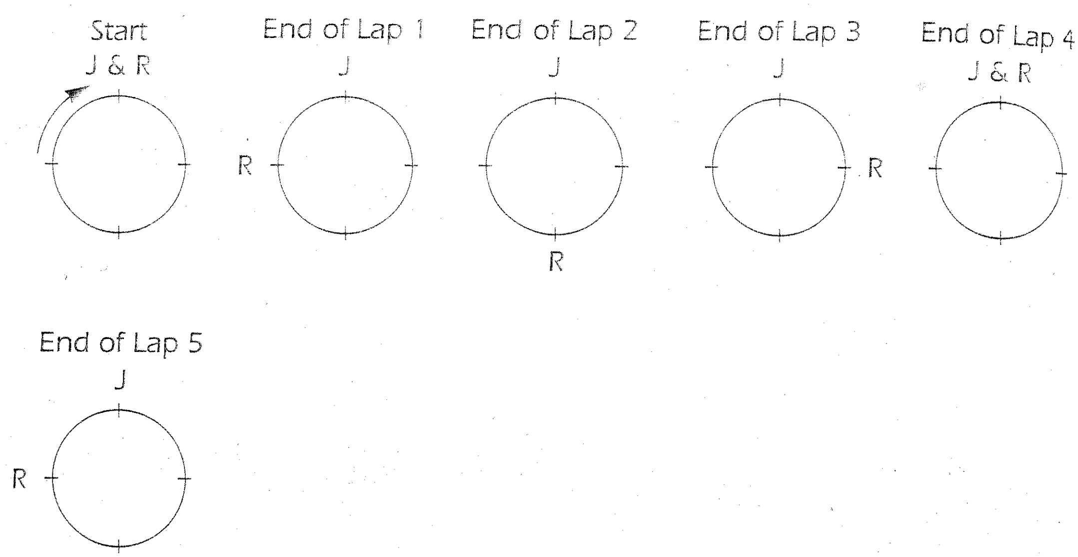
Distance-Rate-Time

Objective: interpret speed as a comparison of distance to time with circular motion.

Not all distance-rate-time problems involve average speed or forward motion. Sometimes we travel in circles—on a track or on a designated loop that brings us back to the starting position. Jeff and Roberta also bicycle on a short track. Jeff can make it around the track in 3 minutes; it takes Roberta 4 minutes. They start at the same place and decide to race around the track 5 times. How long does the race take? Who wins? Are they ever at the same spot along the track at the same moment? If so, at what time and where?

Things to Think About

In this problem we are not given any information about the speed or rate of the riders but we can draw some conclusions. First, they are traveling the same distance, as they are going around the same track 5 times. Second, Jeff must be traveling at a faster speed, since it only takes him 3 minutes to bike around the track versus Roberta's 4 minutes. A picture can help us visualize the relationships. Every time Jeff does a complete lap (in 3 minutes), Roberta has completed $\frac{3}{4}$ lap since in 3 minutes she has gone $\frac{3}{4}$ of 4 minutes.



A ratio table can also be used to keep track of the information.

MINUTES	3	6	9	12	15
LAPS: JEFF	1	2	3	4	5
LAPS: ROBERTA	$\frac{3}{4}$	$1\frac{1}{2}$	$2\frac{1}{4}$	3	$3\frac{3}{4}$

It takes 15 minutes for Jeff to complete the 5 laps. The two riders meet at the starting point after 12 minutes when Jeff has completed 4 complete laps and Roberta has completed 3 complete laps, but Jeff than passes Roberta to win the race in 15 minutes. How long does it take Roberta to finish? It takes her 20 minutes (can you explain why?). Since Roberta bikes at a slower speed than Jeff, it will take her a longer time to cover the same distance. Jeff, biking at a faster speed, takes less time to cover the same distance.

Distance-rate-time situations are proportional situations, since the comparison of distance to time is a rate. Did you notice that when distance is a constant amount (5 times around the track), rate and time vary inversely? This means that when a biker's rate goes up or increases, the time needed to complete the race goes down or decreases; but if a biker's speed or rate is slower (like Roberta's) than the time needed to complete the same course has to increase.

Opportunities to explore relationships such as these informally help prepare students for the more formal study of proportionality in middle and high school algebra. ▲

Teaching Ratio Concepts

Ratios and rates are just two of the interconnected topics that contribute to students' ability to reason proportionally. Many researchers suggest that students should be introduced to these topics earlier, perhaps in fourth grade, with the focus on reasoning (developing ratio sense) rather than on formal procedures. At the very least, we should try to blend instruction of fractions, decimals, percents, and ratios together in middle school; if we regularly facilitate discussions about the similarities, differences, meanings within contexts, and strategies and solutions when solving problems involving ratios, we will help students build understanding over time. It is important to realize that much time and many experiences are needed for students to build up a web of knowledge, since these ideas are mathematically complex. Introducing students to ways of thinking multiplicatively that use part-to-part reasoning as well as part-to-whole comparisons and using models such as ratio tables that facilitate the use of multiplicative comparisons are essential. While as teachers we want to strive for precision in language and classification, our more important job is to help students make sense of this complex topic.

Questions for Discussion

1. What is the difference between additive and multiplicative thinking? When do you think students start to reason multiplicatively? What might we do to facilitate this type of thinking?
2. Generate concepts about ratios, and situations and examples where ratios and rates are used. Check the newspaper, Internet, and other resources for ideas. Draw a concept map that connects the topics you listed.
3. How are perimeters, areas, and volumes of similar figures related to the perimeter, area, and volume of the original figure? Describe the important features of a lesson you would use to introduce the idea of scale factor to sixth graders.
4. One method for solving proportion problems is the cross product method. Why does this method work? What are the disadvantages of showing this method to students prior to their developing understanding of proportional situations?