Teaching Ratio and Proportion in the Middle Grades

PROPORTIONAL reasoning is one of the most fundamental topics in middle grades mathematics. Students' ability to reason proportionally affects their understanding of fractions and measurement in elementary school, and it supports their understanding of functions and algebra in middle school and beyond. Given the importance of ratio and proportion, it is typical to see extensive class time devoted to the topic in upper elementary and middle school grades. This research brief addresses the research on what it means to understand ratio and proportion and how teachers can best support that understanding.

What Does It Mean to Understand Ratios?

Reasoning with ratios begins with learning to attend to two quantities simultaneously. Quantities are things that can be measured, such as length, height, distance, time, temperature, steepness, and speed. Thompson (2010) describes quantities as people's conceptions of measurable attributes of objects or events; quantities are the result of thinking about an attribute, such as height, and then finding a way to measure it with a magnitude and a unit, such as 6 feet. Many quantities are ones that can be measured directly, such as length or distance; researchers call these extensive quantities (Piaget, 1952). Other quantities, such as speed or taste, can be measured as a relation between two variables (Nunes, Desli, & Bell, 2003; Piaget & Inhelder, 1975); these quantities are called intensive quantities. For instance, imagine a situation in which you mix a tablespoon of orange concentrate with a cup of water. The sweetness of the resulting mixture is an intensive quantity that you can measure as a ratio of orange concentrate to water.

Attending to two quantities at the same time is a critical step in learning how to reason with ratios, but it can be difficult for children. Before students can reason simultaneously with two quantities, they often reason with just one quantity at a time (Noelting, 1980). For instance, a common question about the orange concentrate scenario addresses whether two concentrates taste the same or different (Lobato & Ellis, 2010): "Does a batch of orange juice made with 2 cans of orange concentrate and 3 cans of water taste equally orangey, more orangey, or less orangey than a batch made with 4 cans of orange concentrate and 6 cans of water?" A typical student response is, "The second batch is more orangey because you used more orange concentrate." This type of response shows that the student may not be able to simultaneously hold in her mind both quantities, the amount of water and the amount of orange concentrate. Students need explicit practice with learning how to think about two quantities rather than just one.

One way to form a ratio is to create a multiplicative comparison of two quantities. For instance, consider a short green piece of rope that is 10 inches long and a longer blue piece of rope that is 25 inches long. You can either ask how much longer the blue rope is than the green rope (15 inches), or you can ask how many times longer is the blue rope (2.5 times as long). The first way of comparing the lengths is an additive comparison; the second way is a multiplicative comparison. An additive comparison is not a ratio, but a multiplicative comparison is (Kaput & Maxwell-West, 1994; Thompson, 1994).

Another way to form a ratio is to compose two quantities to create a new unit, called a composed unit (Lamon, 1994). For example, say a math classroom has 3 girls for every 2 boys. A composed unit would be a 3:2 unit, which could then be iterated (repeated) and partitioned (broken into equal-size parts) to create other equivalent ratios. You could reason that the classroom could have 6 girls and 4 boys, or 9 girls and 6 boys, or 12 girls and 8 boys, all equivalent ratios formed by iterating the 3:2 unit one, two, and three times, respectively. Forming a ratio as a composed unit is a rudimentary concept, which some refer to as *pre-ratio reasoning* (Lesh, Post, & Behr, 1988). However, it is also a very important way of reasoning, because it can be used in conjunction with other concepts to develop an understanding of proportionality.

What Is Proportional Reasoning?

A proportion is a relationship of equality between two ratios (Lobato & Ellis, 2010). In order to reason proportionally, students must not only reason with ratios, but they must also understand that the ratio of two quantities remains constant even as the corresponding values of the quantities change. The ability to build on the idea of the equality of two ratios is a central hallmark of proportional reasoning. Proportional reasoning involves understanding that (a) equivalent ratios can be created by iterating and/or partitioning a composed unit, and (b) if one quantity in a ratio is multiplied or divided by a factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship.

At beginning levels of proportional reasoning, students

can iterate and partition a composed unit to make a family of equivalent ratios. Consider a problem that encourages iteration and partitioning by asking students to generate same-speed pairs (Lobato & Ellis, 2010). Computer simulations such as SimCalc MathWorlds (Roschelle & Kaput, 1996) show different characters, such as a Frog and a Clown, walking across the screen. Students can input different distances and times to experiment with making the Clown character walk faster or slower:

> Frog walks 10 centimeters in 4 seconds. Find as many different ways to make Clown walk the same speed as Frog as you can.

A composed unit would be the joining of 10 cm and 4 seconds to make a 10 cm:4 s unit. Students can iterate the unite to create many other same-scale units, such as 20 cm:8 s, 50 cm:20 s, 100 cm:40 s, and so forth. Students can also partition the 10 cm:4 s unit to create same-scale units such as 5 cm:2 s, 2.5 cm:1 s, and 1 cm:0.4 s.

It is easy to overestimate students' proportional reasoning abilities, especially if the only problems students encounter are ones with easy numbers. Students tend to develop basic iterating and partitioning, such as doubling and halving, early on (Empson & Knudsen, 2003; Way, 2003). However, students will need additional practice with a variety of different problems before they are able to develop more sophisticated iterating and partitioning strategies. For instance, with the same speed problem, say students have to determine how many centimeters Clown could walk in 5 seconds. This is a more difficult task due to the small difference between 4 seconds and 5 seconds. However, it is possible for students to combine iterating and partitioning to determine the answer. They can partition the 10:4 unit into 4 equal parts to get an equivalent ratio of 2.5 cm:1 s. Then students could iterate that ratio 5 times to get 12.5 cm: 5 s.

In time, students will be able to truncate the work of iterating and partitioning composed units by using multiplication. For instance, instead of iterating 5 times as above, one could multiply the ratio 2.5:1 by 5. A key aspect of developing this understanding lies in giving students repeated experiences that prompt them to reflect on the number of groups they have formed as a result of iterating (Lobato & Ellis, 2010). By reflecting on their repeated practice, students can generalize their understanding to know that if one quantity is multiplied by a factor b, then the other quantity must also be multiplied by the same factor in order to maintain the proportional relationship. The same will also hold true for division.

Although composed units are typically easier for middle grades students to begin forming ratios, it is also important that they learn to form multiplicative comparisons and understand how they are connected to composed units. In the above example, we can think about the 2.5 cm:1 s composed unit as a multiplicative comparison as well: the number of centimeters Clown walks will always be 2.5 times as many as the number of seconds he takes. Using multiplicative comparisons is a powerful strategy for proportional reasoning. For instance, in order to solve the question of how many centimeters Clown can walk in 5 seconds, students can think about the original ratio, 10 cm:4 s, as a multiplicative comparison in which the centimeters are always 2.5 times the seconds. Therefore, in 5 seconds he can walk 5 × 2.5 centimeters, or 12.5 cm.

What Is the Role of Ratio and Proportion in Success in Algebra?

Cai and Sun (2002) explain that "proportional relationships provide a powerful means for students to develop algebraic thinking and function sense" (p. 195). In particular, understanding linear function is directly related to reasoning proportionally. A linear equation of the form $y = mx$ can be seen as a statement of proportionality, in which m is the invariant ratio. Karplus, Pulos, and Stage (1983) even defined proportional reasoning in terms of function understanding, explaining that it is reasoning in a system of two variables between which there exists a linear functional relationship.

Understanding a linear equation $y = mx$ as a proportional statement means that the slope, m, can be seen as the constant rate of change in one quantity relative to the change in the other quantity. Students with a solid foundation in proportional reasoning will be better poised to understand the meaning of slope, and they will be less prone to errors stemming from, for instance, calculating slopes from graphs with nonstandard measurements on the axes (Lobato & Thanheiser, 2002; Lobato, Ellis, & Muñoz, 2003). Students will also be able to understand the graph of a line as a collection of points representing infinitely many equivalent ratios.

A strong basis in proportional reasoning will also be able to support an understanding of linear equations in the form $y = mx + b$ as a statement of proportionality, represented by $y = mx$, combined with a vertical translation (b). Students will then be better prepared to understand important connections and differences between $y = mx$ and $y = mx + b$.

Proportional reasoning is also central to an understanding of measurement, which is important for success in algebra and beyond (Lehrer, 2003). Thompson and Saldanha (2003) point to the value of thinking of a measurement as a ratio comparison rather than simply a "number of things." For instance, one can think of the green 10-inch rope as 10 inches,

or 10 little lengths. One can also see the 10-inch rope as a ratio, in which the standard unit of measurement is 1 inch, and the green rope is 10 times as long as the 1-inch unit. This way of thinking can be helpful in situations in which the unit of measurement changes. For instance, say the unit is changed from inches to feet. This requires recognizing that the ratio of 1 inch to 1 foot is 1:12, which is invariant across the change in units. Thus, each inch is 1/12th of a foot. Because the rope is 10 inches long, it will be 10•(1/12), or 5/6 of a foot long.

What happens when understanding measurements in terms of ratios is absent? Thompson (1994) described a fifth grader's response to the question of whether the speed of a car could be measured in miles per century. The student replied, "No, because you would die, or the car would rust away before a century" (p. 179). If the student could conceive of the measure as a ratio comparison, he or she would be better prepared to make sense of the change in units, regardless of the extremity of the change. A measure of speed such as "miles per century" would be just as valid as "miles per hour," because one could construct a ratio between hours and centuries in order to think about a car's speed without having to drive for the duration of the time span.

How Should Teachers Balance Teaching Skills versus Concepts?

Many curricula emphasize procedures and skills for solving proportions, but researchers caution that the most important challenge of developing students' capacity to reason proportionally is to teach ideas and to restrain the quick path to computation (Smith, 2002). Both skills and concepts are important. Teaching skills alone does not guarantee that students will genuinely understand proportional phenomena, especially because students are skilled at mimicking procedures without understanding. For instance, consider Bonita's work in determining how much water would drip from a leaky faucet in 4 minutes, given that it dripped at a steady rate of 6 ounces in 8 minutes (Lobato & Ellis, 2010):

Bonita set up a proportion and used cross multiplication in order to arrive at a correct response of 3 ounces. However, when questioned further, Bonita was not able to explain why the answer should be 3 ounces; she was not yet able to iterate or partition by halving or doubling. In fact, although Bonita could successfully make use of an algorithm to solve some problems, she had not constructed a ratio either as a multiplicative comparison or as a composed unit. This meant that when Bonita encountered a nonstandard problem, she could not correctly solve it. For instance, Bonita was unable to figure out whether one faucet dripping 6 ounces in 20 minutes was dripping slower, faster, or at the same pace as a second facet dripping 3 ounces in 10 minutes. When faced with a problem that was not framed with standard language providing cues for how to set up a proportion, Bonita struggled to make sense of it.

Bonita's experience is not unusual. Proportional reasoning has been shown to be particularly difficult for students who do not understand what is actually meant by a particular proportional situation or why a given solution strategy works (Cramer & Post, 1993; Lesh, Post, & Behr, 1988). Students' thinking in situations that adults see as proportional often shows that they do not understand what is really happening in those situations. In their rush to compute an answer, students may not stop to think about how the quantities in the situation are related (Smith, 2002). The work of forming a ratio and reasoning proportionally is first and foremost a cognitive task, not an algorithm or a procedure.

There is evidence that students do not naturally approach proportional situations the way adults do; they may not "see" the proportionality in a situation, and it takes work to shift from additive comparisons to multiplicative comparisons (Harel et al., 1994; Hart, 1988). Without a foundation in conceptual understanding of ratios, students may then be poised to make mistakes based on cues in the problem. For instance, students may assume that a situation is proportional if a problem gives three numbers with one missing or if the problem involves key words, such as "per" or "rate". However, many problems may contain these cues but not actually represent proportional situations. For instance, consider the following problem from Lamon (1999, p. 223):

> Bob and Marty run laps together because they both run at the same speed. Today, Marty started running before Bob came out of the locker room. Marty had run 6 laps by the time Bob ran 3. How many laps had Marty run by the time Bob had run 12?

Using a proportion would result in an incorrect answer of 24 laps rather than 15 laps. Conversely, situations can be pro-

portional but students may fail to recognize a situation's proportionality if they are focused on finding key words or other cues. One possible reason Bonita was at a loss to answer the question about the two dripping faucets was because it did not fall into the typical format with three given numbers and one missing number.

## Should We Teach Students the Cross-multiplication Procedure?

There are some advantages to the cross-multiplication procedure. It is efficient and widely applicable across contexts and domains. However, research has shown that students either do not easily learn the cross-multiplication algorithm, or they resist using it when they do (Lamon, 1993; Kaput & West, 1994). This may be due to the difficulty of linking the cross-multiplication algorithm to their earlier understanding of ratios (Smith, 2002). The procedure does not match the mental operations involved in the building up strategy, and the cross products lack meaning in any given situation.

For instance, consider the following task: The label on a box of cookies says that the calories per serving are 210 calories. A serving contains 3 cookies. How many calories are in 5 cookies? It is possible to set up a proportion to solve this problem:

$$\frac{210 \text{ calories}}{3 \text{ cookies}} = \frac{x \text{ calories}}{5 \text{ cookies}}$$

When cross multiplying, you get the equation $3x = 210 \times 5$. What is the unit for 210×5? It is not calories per cookie, and calorie-cookie is not meaningful within the context of the problem.

If a student has not yet mentally formed a ratio, either as a multiplicative comparison or as a composed unit, she may not understand what a proportion in the cross-multiplication procedure represents. Instead, a student may potentially interpret the proportion simply as a template for inserting whole numbers into boxes (Lobato & Ellis, 2010):
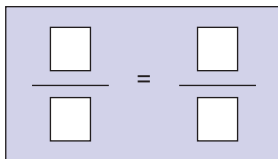
Researchers have found that students often engage in more sophisticated reasoning when not using the proportion algorithm, and that the algorithm can obscure or even interfere with students' understanding of proportionality (Lamon, 2007; Singh, 2010).

The question may not be whether the proportion algorithm should be taught so much as when; if students are not yet reasoning proportionally and have not yet formed a ratio, then the proportion algorithm could be harmful to their understanding (Ohlsson & Rees, 1991; Smith, 2002). However, students who have already gained experience with (a) learning how to simultaneously attend to two quantities, (b) comparing quantities multiplicatively rather than additively, (c) forming a ratio, either as a multiplicative comparison or as a composed unit, and (d) understanding a proportion as an equivalence of ratios, may be poised to understand the value and efficiency of the proportion algorithm.

By the time students reach middle school, they have seen many proportions under the guise of equivalent fractions (Weinberg, 2002). In pre-algebra, students are taught to see a standard proportion algorithm setup as an equation, and they have learned to solve the equation by multiplying both sides by the same number to isolate x. Some researchers have discussed introducing the ideas of multiplication, division, and ratio simultaneously through equipartitioning operations (Confrey, 1995; Confrey & Scarano, 1995); within this approach, fractions are considered a particular subset of ratio relations (Confrey, 2012). The cross-multiplication algorithm is yet another approach, and helping students see the connections across different procedural methods will help them better understand why each procedure works and when it might be the most helpful to use any given procedure.

Providing students with the opportunity to engage in repeated reasoning (Harel, 2007) by thinking through the logic of ratio and proportion problems again and again will help them generalize their understanding into broadly applicable algorithms. For instance, returning to the cookie problem, consider an alternate solution. A student might reason that the ratio of calories to cookies is 210:3. One could either divide or partition to obtain a unit ratio of 70 calories per cookie. Because 5 cookies have 5 times as many calories as 1 cookie, multiply 70 by 5 to get 350 calories. This approach relies on the same idea as the cross-multiplication algorithm, but it is grounded in sense making. Engaging in repeated reasoning of concepts, rather than just repeated practice of procedures and skills, can foster both understanding and skill development for students. Ultimately, students should then be encouraged to formalize their repeated reasoning into a general procedure that can be applied to many different problems.

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