Building

Proportional-Reaso

MELANIE PARKER

THE CONCEPTUAL DOMAIN OF RATIO AND proportion may be a bridge that permits access to higher-level thinking in mathematics (Lesh, Post, and Behr 1988). Future teachers who have developed a variety of rich representations and flexible ways of thinking about proportional relationships are well positioned to offer instruction that helps students move from using informal strategies to expressing proportional relationships in algebraic terms. I have found that exploring informal "building up" strategies with preservice teachers leads nicely to formalizing proportional relationships algebraically. In this article, I first share what I mean by "building up" strategies. A description and discussion of selected activities that have been used successfully with preservice teachers follow.

What Are "Building Up" Strategies?

CONSISTENT WITH THE VIEW THAT STUDENTS create meaning for mathematical ideas and computations for themselves, recent research has revealed proportional-reasoning strategies used by middle school students as they engage in oral problem-solving sessions (e.g., Lamon [1994]). Students who have not yet encountered the formal study of proportions, for example, setting up and solving a missing-value proportion, frequently use an additive building-up strategy to solve problems. The use of an additive strategy can be a natural transition to a multiplicative strategy, since multiplication can be used to produce the same result as repeated addition. **Figures 1** and **2** illustrate these common building-up strategies. **Figure 1** uses simple additive incremental steps, and **figure 2** uses a picture

MELANIE PARKER, *parker@mail.clarion.edu, teaches mathematics and mathematics education courses at Clarion University of Pennsylvania, Clarion, PA 16214-1232. Her interests include the teaching and learning of percent and future teachers' understandings of multiplicative structures.*

on “Building Up”:

ning Activities for Future Teachers

of the groupings, which leads to a solution by multiplication for the following problem.

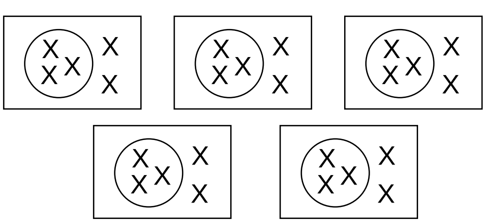
> In a class of twenty-five students, three out of every five students ride the bus to school. How many students ride the bus?

The strategy in **figure 1** is purely additive; the strategy in **figure 2** is more sophisticated because it uses the multiplicative relationship inherent in this situation.

	Bus Riders	Class Members
	3	out of 5
so	6	out of 10
so	9	out of 15
so	12	out of 20
and	15	out of 25

So there are 15 bus riders.

Fig. 1 An additive building-up strategy



The twenty-five students are arranged in groups of five. Three of each group of five students are circled, so 3 • 5 = 15 ride the bus.

Fig. 2 A multiplicative building-up strategy

An interesting example of the application of a building-up strategy concerns the Scavo and Conroy (1996) discussion of two alternative solutions, one algebraic, one “bizarre” but producing the desired result, to the following problem presented in a first-year-algebra class (p. 684):

> Two numbers are in the ratio of 2 to 5. One of the numbers is 21 more than the other. What are the two numbers?

One student’s formal algebraic solution involved the solution of two simultaneous equations,

(1) $$\frac{x}{y} = \frac{2}{5}$$

and

(2) $$y = x + 21.$$

A second student’s solution involved the following series of simple computations:

$$5 - 2 = 3$$

$$\frac{21}{3} = 7$$

$$2 \bullet 7 = 14 \ (= x)$$

$$5 \bullet 7 = 35 \ (= y)$$

Intrigued by the fact that this unusual series of computations *always* produces the correct result, and concerned by the frequent use of non–conceptually based mathematical “tricks” that sometimes seem to be accepted as legitimate processes, Scavo and Conroy were able to provide an algebraic justification accessible to eighth graders. The proof that they presented is elegant and requires a reformulation of the proportions and substitution. Although the given proof verifies that the student’s series of computations

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works every time, the application of building-up strategies may offer another explanation for the second student's series of computations.

First, consider that this problem can be solved with an additive building-up strategy, such as the one that follows. This solution simultaneously "builds up" and keeps track of the differences between the pairs of numbers.

NUMBERS	DIFFERENCE
2 and 5	3
4 and 10	6
6 and 15	9
8 and 20	12
10 and 25	15
12 and 30	18
14 and 35	21

So the two numbers are 14 and 35.

Scavo and Conroy's second student may have solved the problem by using a multiplicative variation on the foregoing approach. By noting that the given difference of 21 is obtained by increasing the difference ($5 - 2 = 3$) a total of seven times ($21/3 = 7$), the student may have concluded that seven 2s and seven 5s would be needed to produce the desired numbers ($2 \bullet 7 = 14$, $5 \bullet 7 = 35$). Central to this strategy is determining that the multiplicative constant of 7 will transform the original numbers (2 and 5) into the desired numbers (14 and 35):

Written work cannot always give a picture of one's knowledge

$$\begin{array}{ccc} 2 & : & 5 \\ \times 7 \downarrow & & \downarrow \times 7 \\ 14 & : & 35 \end{array}$$

As illustrated by Scavo and Conroy's second student's response, written work alone cannot always give a full and accurate picture of a person's knowledge of mathematical concepts. We can learn a great deal more about a person's thinking by also asking him or her what was done and by listening carefully to discourse in group activity. So interviews and group activities, which have proved to be exceptionally revealing of future teachers' knowledge of mathematics (e.g., Ball [1990]; Parker [1994]), are fundamental to the implementation of similar problems that I use with preservice teachers.

Activities for Future Teachers

THE CONCEPTUAL ROOTS OF RATIO CAN BE UNcovered in "sharing" activities, both in the classroom and at the level of preservice teacher education (Parker and Leinhardt 1995). In the study of ratio in the mathematics content course that I teach for prospective elementary teachers, I begin with a hands-on sharing activity. Pairs of preservice teachers are given a cup of corn kernels. They are to share the kernels, using the following "unfair" sharing rule: you get 3 and I get 5. These students are to continue dealing out the kernels until one has 12 and the other has 20. I then ask them to determine the total number of kernels that would be dealt if the process continued until one person had 100 kernels more than the other. Within the course, I encourage my students to use alternative problem-solving strategies and to engage in novel problems. They are not told that the activity involves proportional relationships, so the solution strategies that they employ seldom include setting up a missing-value proportion. My intent is to strengthen and extend the conceptual foundation of proportional relationships already held by these future teachers and to help them formalize their solution procedures in algebraic terms.

Using the strategies "do a simulation," "make a list," and "look for a pattern," the preservice teachers keep track of the number of kernels each has after repeated "deals." Once a solution has been obtained, each pair presents its solution method to the entire group. The most commonly used methods are like the procedures discussed previously: using a building-up strategy to investigate patterns and then using the pattern to discover the multiplicative constant that will transform 3 and 5 into numbers that differ by 100. The type of discussion that often occurs follows and offers a wonderful opportunity to reinforce the interpretation of multiplication as repeated addition.

DEAL	NUMBERS	DIFFERENCE
1	3 : 5	2
2	6 : 10	4
3	9 : 15	6
4	12 : 20	8

Each time we "deal," the difference between our amounts increases by 2. We will be 100 apart after 50 deals (the multiplicative constant).

Therefore, you have 3 • 50 = 150 and I have 5 • 50 = 250; that is, 400 kernels have been dealt.

At this point, my students and I spend some time expressing this idea in algebraic terms.

The ratio can be expressed as $3x : 5x$, for any nonzero x. The difference between the two amounts is $5x - 3x$, or $2x$. The given difference is 100, so

$$2x = 100,$$

thus

$$x = 50.$$

Therefore, the ratio $(3x : 5x)$ can be expressed as 150 : 250, with a difference of 100 between the two values.

A second problem is then posed. The preservice teachers are asked to determine how many kernels each has when a total of 480 kernels have been distributed. A common solution follows:

On each "deal," 8 kernels are distributed. Therefore, 480/8, or 60, deals are necessary to give out 480 kernels. So, you have 3 • 60 = 180 and I have 5 • 60 = 300.

Formalizing this solution gives the following algebraic form:

x = the number of deals.
$3x + 5x = 8x$, the number of kernels dealt in x deals.
$8x = 480$.
$x = 60$ deals.
$3x = 3(60) = 180$, the number that you have.
$5x = 5(60) = 300$, the number that I have.

These types of activities are designed to help future teachers develop flexible ways of thinking by involving them in the development of multiple, rich representations of proportional relationships. As we explore the concept of ratio and proportion, I seek to have them understand both the additive and multiplicative interpretations of the building-up strategy, as well as the algebraic representations. In this way, those who have different types and levels of understanding about proportional reasoning are able to see various ways to approach the problems and to grapple with the relationship between their strategies and other, sometimes more sophisticated, methods.

Closing Comments

IF FUTURE TEACHERS—AND, SUBSEQUENTLY, their students—are to develop rich proportional-reasoning abilities, they must be encouraged to make sense of proportional situations at all levels: informal and formal. Alternative solution methods should be encouraged and valued, discussed in class, and connected to one another. To make sense of the algebra, connections between the abstract algebraic solution and the simpler, more intuitive, solutions for problems need to be made explicit. To create this environment for prospective teachers, and potentially for their students, teachers at all levels need to listen carefully to their students and to encourage students to listen to one another, discussing whether arguments, both formal and informal, are valid and mathematically sound. In this way, both teachers and students can build their understanding of proportional reasoning.

The activities help future teachers develop flexible ways of thinking

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