

ONE-HOUR WORKSHOPS



YOU ARE INVITED TO THE FOLLOWING ONE-HOUR WORKSHOPS, AT THE NCTM CONFERENCE IN WASHINGTON DC

Algebra for All in the Middle Grades

HP Calculators has a new solution for middle grades math. Come get hands-on experience with the easy-to-use HP 39gs graphing calculator and the StreamSmart data-streamer! Motivating students to learn Algebra has never been easier!

Thursday, April 23, Room 143C (Convention Center) – Exhibitor Workshop #3 8:30 a.m. – 9:30 a.m.

Thursday, April 23, Room 143C – Exhibitor Workshop #15 1:00 p.m. – 2:00 p.m.

Friday, April 24, Room 143C – Exhibitor Workshop 31 10:00 a.m. – 11:00 a.m.

A New CAS Graphing Calculator for AP Calculus and Statistics

Introducing the HP 40gs graphing calculator with its interactive CAS for AP Calculus and both descriptive and inferential statistics applications for AP Statistics. Come get hands-on experience with the HP 40gs and the StreamSmart data-streamer.

Friday, April 24, Room 143C – Exhibitor Workshop 39 1:00 p.m. – 2:00 p.m.

PARTICIPANTS HAVE A CHANCE TO WIN ONE OF 10 SETS OF HP MOBILE CALCULATING LABS (MCL) PER WORKSHOP.

Each set includes the following:

- 1 HP 39gs Graphing Calculator
- 1 HP StreamSmart 400 data collector
- 5 Fourier sensors (distance, temperature, light, gas pressure, and microphone)
- 1 Virtual HP 39gs Graphing Calculator software application for the PC
- 1 Middle Grades Mathematics Activity Book with activities, teacher guides, and answer keys

Email us at **Calceducation@hp.com** or visit **www.hp.com/calculators** for more product information.

Giant Developing Proportional Reasoning

S. Megan Che

Teachers can help students develop proportional reasoning and explore measurement with an open-ended activity.

404 MATHEMATICS TEACHING IN THE MIDDLE SCHOOL • Vol. 14, No. 7, March 2009

Copyright © 2009 The National Council of Teachers of Mathematics, Inc. www.nctm.org. All rights reserved. This material may not be copied or distributed electronically or in any other format without written permission from NCTM.



On the first day of algebra class, I would often begin by posing a problem that focused on proportional reasoning, an important mathematical concept in the middle grades (NCTM 2000). Taped to the walls of the classroom were drawings of large pencils approximately 20 inches long and 5 inches wide—and the students were presented with the following scenario:

Posted around the room are four large pencils that a giant might use during mathematics class. If the giant uses pencils of this size, what can you find out about the giant?

Some questions you might pose to students could include these:

- How tall is the giant?
- How much does the giant weigh?
- How many times larger than you is the giant?
- Could the giant walk in the door? Could he stand upright in the room?
- How realistic is the pencil shape? Is it too wide compared with our pencils? Is it too long? Is the eraser an appropriate size?
- How big would the giant's desk need to be?
- If you eat 2000 calories each day, how many calories might the giant need to eat?

This is not an exhaustive list of possible questions that you can ask.

The students in my algebra classes generally followed a problem-solving process similar to that outlined by Wheatley and Abshire (2002). A rich mathematical task is posed to students, they think about and work on the problem in pairs, and then strategies are discussed as a class. This process worked well in my classes after students understood why I was not going to give them the answer. In particular, having my students collaborate in pairs on mathematical tasks was fruitful, in part because they were allowed to choose their peer partner. I noticed that students almost always selected a partner who was familiar and at a similar ability level.

The pairs of students began to work immediately. Most of the students had already begun to question their partners, share ideas, and generate strategies collaboratively. The social construction of understanding (von Glasersfeld 1995) had become blurred by the time I was able to circulate around the room and visit the pairs of students. The opportunity to find out more about each student's understanding and reasoning was lost. If I were to present this task again, I would ask students to think about the problem for a few moments on their own before collaborating with a partner.

ADDITIVE STRATEGIES

This task has the potential to challenge students in many ways. One such challenge is that it created a stumbling block for students who were inclined to use addition on this task and encouraged them to reason proportionally. For instance, one pair of students calculated the difference in inches between the lengths of an actual pencil and the giant's pencil, then added this difference to their own height, in feet, to get the giant's height, in feet. Students who used this additive strategy explained that since the giant's pencil was some amount longer than theirs (for example, 8 inches), the giant should be that same amount taller. Instead of seeing the giant's pencil as being, for example, two times longer than their pencil, these student pairs saw the giant's pencil as being 8 inches longer. I did not, however, point out the difference in reasoning to the students at that time. I anticipated that these student pairs, when using this strategy, would question the giant's weight when it was determined.

When these students added 8 feet to their height to approximate the giant's height and began to think about what the giant's weight might be, they were faced with a dilemma. They realized that adding 8 pounds to their weight to get the giant's weight would not make sense for such a large person. If a person was 8 feet taller than they, that person would weigh more





S. Megan Che, sche@clemson.edu, is an assistant professor of mathematics education at Clemson University in Clemson, South Carolina. She is interested in the topics of rational numbers, proportional reasoning, and equity in mathematics. She is currently involved in research on gender and mathematics learning.

than 8 additional pounds. When confronted with this flaw in their additive strategy, these students then conferred with another pair that had adopted a multiplicative strategy.

Those using a multiplicative approach explained that they found how many pencils would fit into the giant pencil by stacking them up until the width of the big pencil was covered. They found that about 7 1/2 pencils were needed to fill the width of 1 big pencil. Then they used one student's height and multiplied it by 7 1/2 to get a prediction for the giant's height. The students who had been using an additive strategy for their predictions adopted this method of finding a scale factor and multiplying because, as they said, their results would make more sense and they would no longer have to improvise using units.

MEASUREMENT ISSUES

Students who began using additive strategies faced a problem with their units of measure, because the units in their answers did not seem realistic. Most students initially dismissed or ignored this issue. Although ignoring unrealistic measurements was both alarming and problematic for me, most students showed and voiced no concern as they measured pencils in inches and then added height in feet. One pair's reasoning typifies the explanations of students using this additive strategy. They changed their units from inches to feet when adding to the height because 8 inches of height would not be enough to account for such a large pencil; they knew people who might be 8 inches taller than they, but these people did not need such big pencils. These students must have been uncertain to some extent about their strategy for dealing with units, because when they conferred with or noticed another group using a proportional

strategy, those using addition were easily converted to a proportional strategy.

This change to using proportion resolved the units problem because scale factors do not have units. It made sense to them that if the giant's pencil was twice as long as their pencil, then the giant might be approximately twice as tall, as well. However, just because the giant's pencil is 8 inches longer than their pencil, it did not make sense that the giant was only 8 inches taller. These statements provide evidence of students' emerging proportional reasoning.

A second measurement issue emerged because the format of the problem task involved two-dimensional representations of three-dimensional objects. I had not brought in enormous three-dimensional toy pencils or crayons but rather had taped two-dimensional images to the wall. Although it added to the complexity of the task, students compared their three-dimensional pencils with the two-dimensional drawings, which opened up their options. They had to decide whether they should stack their pencils to see how many would fit in the giant pencil to compare widths, line up their pencils across to compare lengths, or calculate the relative approximate volumes of pencils for comparison. Students rarely proposed this third option. Some students reasoned that comparing pencil widths made more sense than comparing lengths. The width of a pencil does not change, but the length changes each time it is sharpened. Few students compared eraser lengths for giant and regular pencils for the same reasonthey did not know what portion of the giant eraser might have been used. The metal band wrapped around pencils just below the eraser is a constant size; I will include this metal band in the giant-pencil sketches in the future as another option for relational measurement.

A third measurement issue that arose was degree of accuracy. Most of the students casually rounded their measurements to half or even whole inches, usually with little or no justification for such choices and little or no thought about the consequences of rounding their results. I observed that many students struggled to measure the widths of their pencils with rulers or tape measures, often measuring to the nearest eighth or sixteenth of an inch, then using the very same instrument to measure the width of the giant pencil, rounding to the nearest half inch. As a class, when strategies were compared and contrasted, we discussed the issue of measuring and accuracy because most pairs of students settled on different degrees of accuracy. We talked about what level of accuracy is necessary for this task and how that accuracy level might change depending on the task. Thus, many students began to understand mathematical and practical consequences of rounding.

PROCEDURAL STRATEGIES

Many of the student pairs who correctly used proportions as a strategy had difficulty articulating a rationale for their specific procedure. Most set up a proportion with an unknown and then cross multiplied to solve for the unknown (see **fig. 1**), but they could not communicate why this cross-multiplication procedure worked.

This could have been an occasion to demonstrate algebraically why cross multiplication works by multiplying both sides twice by conveniently chosen quantities. Instead, we took the route of connecting the strategies of those who used scale factors with those who used cross multiplication. To determine the giant's height, for instance, a student pair who had used scale factors might have worked out the problem as explained in **figure 2**.

Ås a class, we explored how these two strategies related. Through dis-

cussions including the meaning and arrangement of different numbers, some students could begin to understand that the cross-multiplication strategy essentially involves multiplying a quantity by a scale factor. I showed the class that the scale factor strategy can be written in a cross-multiplication format if we compare, for example, regular and giant measurements, as explained in **figure 3**.

Once students recognized scale factors in cross-multiplication strategies, they began to understand that this same scale factor can be used as a basis for calculating not just the giant's height but also other characteristics, such as weight, without needing to set up a proportion to solve each time. Further, students could understand why the scale factor was so useful. If they had calculated that for each inch of their pencil there were approximately 2.41 inches of a giant pencil, this unit rate could be a reasonable basis for other units as well. When each inch of a regular pencil compares with 2.41 inches of a giant pencil, it makes sense that each foot of their height might be 2.41 feet of a giant, or even that each of their pounds might be $(2.41)^3$ pounds of a giant.

EXTENSIONS TO THE ACTIVITY

For this task, students typically took about forty-five minutes to construct responses with their partner. Then we shared strategies and results as a class for another thirty to forty-five minutes. Thus, depending on how much time is spent on the student pair problem-solving phase and on the whole-class discussion, it is possible-especially with a block schedule-to reach a resolution of this problem in one class session. However, in my experience, we frequently had to return to the students' results the next day because we could not address all the mathematical issues fully in one day. Students could also

Fig. 1 The student is attempting to find the value for the giant's height with proportion using the ratio length/height.

 $\frac{\text{length}}{\text{height}} : \frac{8.5 \text{ in.}}{5.5 \text{ ft.}} = \frac{20.5 \text{ in.}}{x \text{ ft.}}$ (8.5 in.)(x ft.) = (20.5 in.)(5.5 ft.) $x \text{ ft.} = \frac{(20.5 \text{ in.})(5.5 \text{ ft.})}{8.5 \text{ in.}}$

Fig. 2 The student is attempting to find the value for the giant's height by determining and using a scale factor.

Giant pencil length $= 20.5$ in.	My pencil length $= 8.5$ in.
$\frac{20.5 \text{ in.}}{8.5 \text{ in.}} \approx 2.41, \text{ so my pen}$	cil goes into the giant's pencil 2.41 times.
My height = 5.5 ft. If 2.41 of my heights go into the giant's height, then the giant's height should be 5.5 ft. \times 2.41.	

Fig. 3 Solving the problem with cross multiplication is a way to use a scale factor in computation.

giant measurements regular measurements	$:\frac{x \text{ ft.}}{5.5 \text{ ft.}} = \frac{20.5 \text{ in.}}{8.5 \text{ in.}}$
	= 2.41, the scale factor
	x ft. = 2.41(5.5 ft.) = 13.26 ft.

ponder their strategies overnight. This unexpected turn of events resulted in a smooth segue into the next day of class because some students would start asking questions about the task as they were coming into the room. Not tying up every loose end at the conclusion of each class also aided in our co-construction of classroom culture and sociomathematical norms (Yackel and Cobb 1996).

Many students—eventually and sometimes after much resistance—began to understand that many complex problems take time to think about and make sense of. They also learned that the process of constructing mathematical meaning from these tasks is at least as important as arriving at a result. By presenting this task at the beginning of the year, it sometimes became acceptable to end class without a final answer.

Given this freedom from being compelled to wrap up each interesting mathematical task by the end of class, numerous avenues for extensions arise. By asking questions about how many students, teachers, and giants might be able to fit in the classroom, this task can be connected with spatial reasoning and geometric notions of volume, especially in the absence of yardsticks or metersticks. Posing such questions may also invoke students' thinking about using volume, rather than width or length, of a normal object for comparison with its giant equivalent. Relationships between this task and statistics can be constructed if one focused on the assumptions that

students made about representative pencils and persons to compare with the large pencil and the giant:

- Did students simply use their own heights, for instance, assuming that the giant would be close to their age?
- Did they estimate a mean for pencil dimensions and heights from their group?
- Did they use data from the whole class to decide on values for height and pencil dimension?
- How might these decisions and assumptions relate to students' confidence in the accuracy of their findings?

Another extension that could focus students' attention again on proportional reasoning is the question of whether the drawing of the giant pencil was realistic. We rarely had time in our classes to discuss this question, but it would be beneficial to return to it, especially if the curriculum devoted a large portion of time to the development of proportional reasoning. The proportion of the width to the length of the giant pencil was approximately 5 inches to 20 inches. Would it be realistic for a standard pencil to have similar proportions? How long might a standard pencil need to be to have dimensions of the same proportion as the giant pencil? This extension could

provide a natural bridge to the topic of similar geometric figures. After using this task with two-dimensional drawings of giant pencils, questions could be asked about three-dimensional models, such as wooden dowels or giant toy crayon. Although some students will likely still attempt to compare along one dimension (length, width, or depth), some students may begin to think in terms of filling up the larger crayon with a number of the smaller crayons. Not using a three-dimensional model from the start gives the students the chance to struggle with visualizing a three-dimensional object from a two-dimensional representation. They have the chance to justify their strategy for comparison and think about why they might want to use width rather than length for comparison. With a two-dimensional image, they can speculate about what might happen if they had a three-dimensional model for comparison instead. Then, providing a three-dimensional model would not only give students a chance to try out their hypotheses, it may also challenge them to think even further about appropriate measures of comparison. They may start to consider using length, width, depth, surface area, volume, or some other choice, such as the width of the metal band below the eraser, for comparison.

CONCLUSION

Although many rich proportional reasoning problems are available to middle-grades teachers and students, this particular task can potentially combine the use of multiple strategies, mathematical thinking, and discourse using hands-on measurement. While working in pairs, students can become actively engaged in thinking more concretely about the meaning of ratios and proportions. This task provides specific avenues for proportional reasoning because students physically measure objects and then choose a strategy. The measurements are tangibly present before them as they confront the limits of additive strategies. Mathematical discussions emerging from this giant pencil task can focus on approaches to the problem and the accuracy of results.

According to NCTM's Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics (2006), tasks such as the one presented here that focus on constructing proportional reasoning are ideally suited for the seventh grade. Although I used this task with algebra students, it could be even more powerful when used with students who have not yet been taught the cross-multiplication strategy for working with proportions. This task was conducive to generating mathematical discourse. In my experience, given the numerous avenues for solutions, it can engage all students in mathematical reasoning. In other words, it is a rich mathematical investigation (Wheatley and Reynolds 1999). Students are challenged to think carefully about their decisions and articulate their assumptions as they try various strategies for solving the task. This giant pencil activity is also easily extendable and thus can quickly be adapted as students progress through the task in often different ways and at different rates.

REFERENCES

- National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, VA: NCTM, 2000.
- Von Glasersfeld, Ernst. Radical Constructivism: A Way of Knowing and Learning. Washington, DC: The Falmer Press, 1995.
- Wheatley, Grayson H., and George Abshire. *Developing Mathematical Fluency*. Tallahassee, FL: Mathematics Learning, 2002.
- Wheatley, Grayson H., and Anne M. Reynolds. Coming to Know Number: A Mathematics Activity Resource for Elementary School Teachers. Tallahassee, FL: Mathematics Learning, 1999.
- Yackel, Erna, and Paul Cobb. "Sociomathematical Norms, Argumentation, and Autonomy in Mathematics." *Journal* for Research in Mathematics Education 27 (July 1996): 458–77. ●