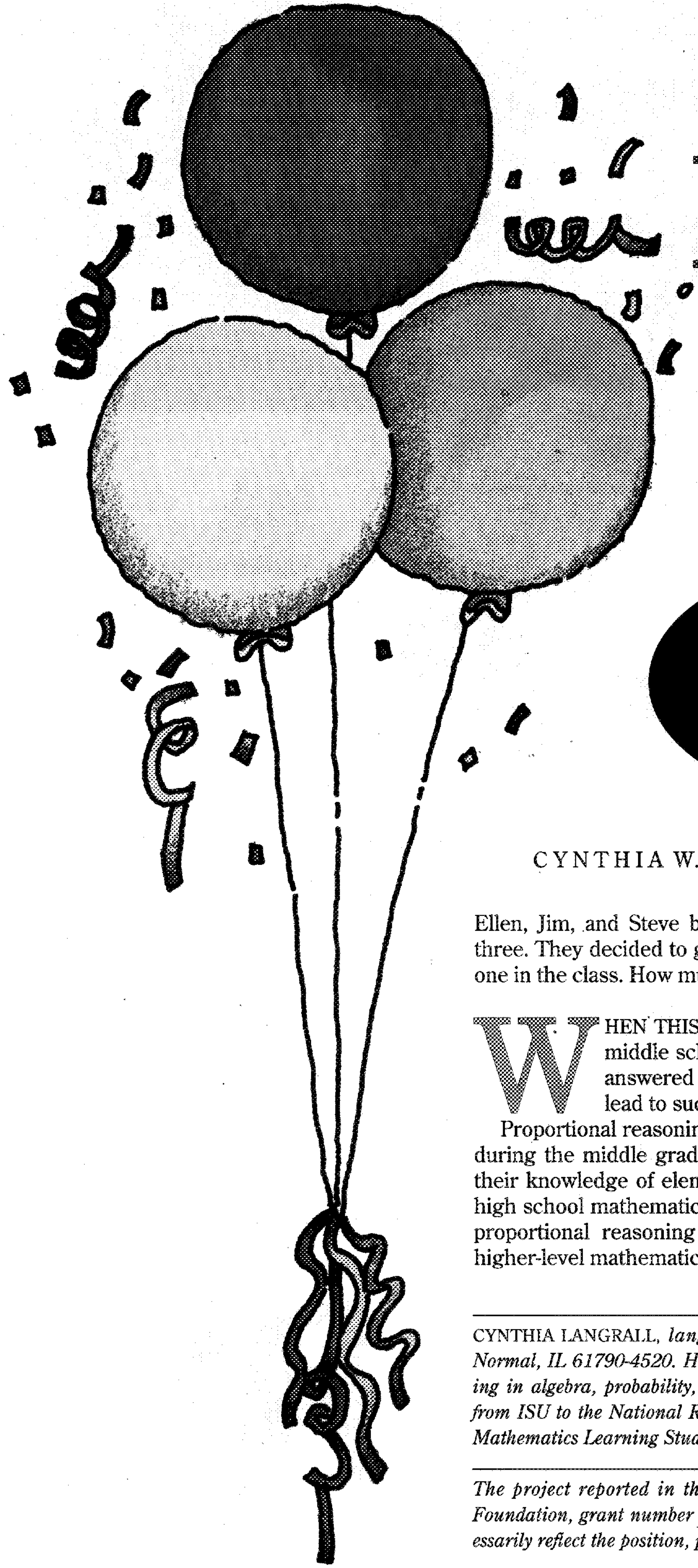

Three Balloons for Two Dollars:

Developing Proportional Reasoning

CYNTHIA W. LANGRALL AND JANE SWAFFORD

Ellen, Jim, and Steve bought three helium-filled balloons and paid $2 for all three. They decided to go back to the store and buy enough balloons for everyone in the class. How much did they pay for 24 balloons? (Lamon 1993b)

WHEN THIS PROPORTION PROBLEM WAS GIVEN TO A GROUP OF middle school students, some correctly answered $16, but others answered $8, $12, $24, and $26. What kind of reasoning would lead to such diverse responses?

Proportional reasoning is one of the most important abilities to be developed during the middle grades. Using proportional reasoning, students consolidate their knowledge of elementary school mathematics and build a foundation for high school mathematics and algebraic reasoning. Students who fail to develop proportional reasoning are likely to encounter obstacles in understanding higher-level mathematics, particularly algebra.

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Through the Peoria Urban Mathematics Plan (PUMP) for Algebra project, a teacher-enhancement project funded by the National Science Foundation, we have been working with forty-four middle school teachers in a midsize urban city to increase the number of students enrolled in algebra. To help teachers understand why many of their students are not successful with proportions, we searched the literature on proportional reasoning to see whether it offered any insights. We drew from the work of Lamon (1993a, 1993b, 1994, 1995) and the Rational Number Project (Cramer, Post, and Currier 1993), incorporating their findings into our work in PUMP classrooms. This article shares the insights that we gained from studying the literature and our experiences using many of the problems found in the literature with our urban middle school students.

Overview of Proportion

A PROPORTION IS THE STATEMENT THAT TWO ratios are equal in the sense that both convey the same relationship. For example, the ratio of three balloons for \$2 is the same as that of twenty-four balloons for \$16. Hence, the statement $3/2 = 24/16$ is a proportion. A proportion expresses a multiplicative relationship between two quantities, in this example, balloons and dollars. The ratio 3/2 conveys this multiplicative relationship, meaning that for every 3 balloons, the cost is \$2, or that for each dollar, 1.5 balloons can be purchased. To find how many balloons you can buy for a given amount of money, you must multiply the amount by 3/2 or 1.5.

Problem Types

LAMON (1993B) IDENTIFIES FOUR DIFFERENT types of proportion problems (see **fig. 1**). In a part-part-whole problem, a subset of a whole is compared with its complement (e.g., boys with girls, correct answers with incorrect answers) or with the whole itself (e.g., 12 boys out of 20 students, 80 correct answers out of 100 questions). Problems that involve associated sets relate two quantities, which are not ordinarily associated, through a problem context, such as balloons and dollars, people and pizza, or cookies and boxes. Problems involving well-known measures express relationships that are well-known entities or rates, such as speed, which is the ratio of miles and hours, or unit price, which is the ratio of items and dollars. Growth problems express a relationship between two continuous quantities, such as height, length, width, or circumference, and involve either *scaling up,* that is, enlarging or stretching, or *scaling down,* that is, reducing or shrinking.

Part-part-whole

Mrs. Jones put her students into groups of 5. Each group had 3 girls. If she has 25 students, how many girls and how many boys does she have in her class?

Associated sets

Ellen, Jim, and Steve bought 3 helium-filled balloons and paid \$2 for all 3 balloons. They decided to go back to the store and buy enough balloons for everyone in the class. How much did they pay for 24 balloons?

Well-known measures

Dr. Day drove 156 miles and used 6 gallons of gasoline. At this rate, can he drive 561 miles on a full tank of 21 gallons of gasoline?

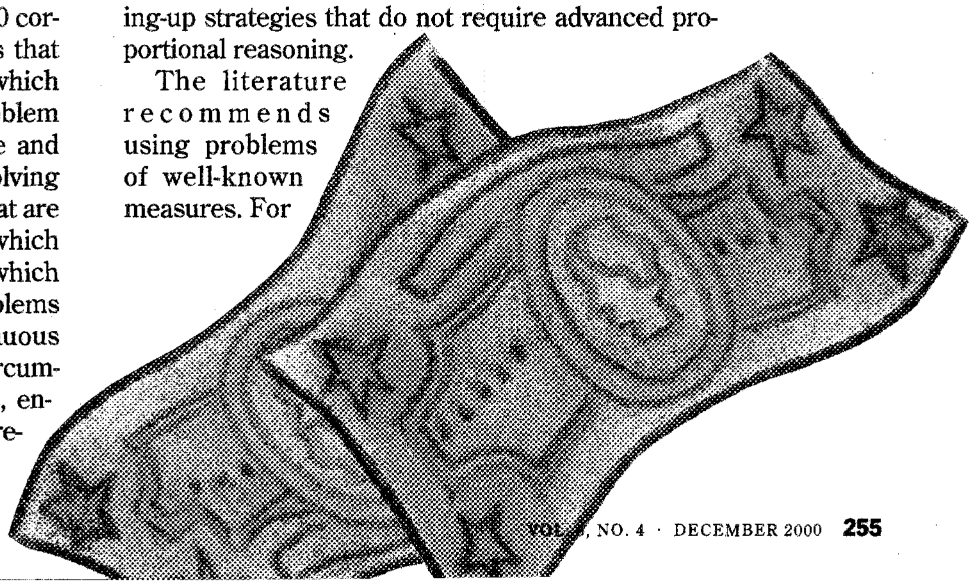
Growth (stretching and shrinking situations)

A 6" × 8" photograph was enlarged so that the width changed from 8" to 12". What is the height of the new photograph?

Fig. 1 Types of proportion problems

Different problem types elicit different solution strategies regardless of a student's level of understanding of proportional reasoning (Lamon 1993b). Research indicates that students tend to use a higher level of proportional reasoning strategies in solving problems of associated sets. The language of ratio is elicited more naturally when students are forced to think about two sets, not typically associated, as a composite that relates one to the other in the context of the problem. With part-part-whole problems, students are inclined to use informal methods of reasoning, even when they have demonstrated higher-level thinking in other problems, for the reason that part-part-whole problems lend themselves to counting, matching, and building-up strategies that do not require advanced proportional reasoning.

The literature recommends using problems of well-known measures. For



some students, familiarity with such well-known measures as speed and price may facilitate proportional thinking; but for others, the familiar language may allow them to mask their lack of understanding (Lamon 1993b). Students who have learned formulas for working "miles per hour" problems, for example, may be able to solve these problems. However, they do not necessarily understand the multiplicative, or proportional, relationship involved in "miles per hour," that is, that a particular number of miles is traveled each hour. In all the literature we reviewed, growth problems were identified as the most difficult type (e.g., Cramer, Post, and Currier [1993]; Lamon [1993b]). Unlike the part-part-whole and associated-sets problems, which involve discrete quantities, growth problems involve continuous quantities, which are more difficult for students to represent with objects or pictures.

Student Solution Strategies

USING PROBLEMS FROM EACH OF THE FOUR CATegories, we interviewed sixteen middle school students from PUMP classrooms to discover what kind of strategies they would use to solve the different types of problems. Four students each from fifth through eighth grade were selected by their teachers for the interviews on the basis of their general levels of mathematics understanding. One student from each of the grades had a low level of understanding, two had average levels, and one had a high level.

Level 0: Nonproportional reasoning

- Guesses or uses visual clues ("It looks like . . .")
- Is unable to recognize multiplicative relationships
- Randomly uses numbers, operations, or strategies
- Is unable to link the two measures

Level 1: Informal reasoning about proportional situations

- Uses pictures, models, or manipulatives to make sense of situations
- Makes qualitative comparisons

Level 2: Quantitative reasoning

- Unitizes or uses composite units
- Finds and uses unit rate
- Identifies or uses scalar factor or table
- Uses equivalent fractions
- Builds up both measures

Level 3: Formal proportional reasoning

- Sets up proportion using variables and solves using cross-product rule or equivalent fractions
- Fully understands the invariant and covariant relationships

Fig. 2 Proportional reasoning strategies

Level 0

From our interpretation of the literature and the results of the interviews, we identified four different levels of strategies for proportional reasoning (see **fig. 2**). Strategies at level 0 involve no proportional reasoning. These strategies are characterized by additive rather than multiplicative comparisons or random use of numbers or operations in the problems. They do not lead to correct solutions or the development of more mature proportional reasoning. Kerry, for example, randomly selected numbers to divide in the Balloon Problem and obtained an answer of $8, as revealed in the following dialogue:

I. How did you determine your answer?

K. Because if they say they paid $2 for all three balloons, then they decided to go back and pay for twenty-four balloons. I took 3 divided by 24.

I. Why did you divide 24 by 3?

D. Because when they said how much did they pay for twenty-four balloons, I thought of something you could divide by 24.

Although Kerry may have had a reason for choosing division as her operation, her choice of the numbers to divide was arbitrary and not justified by the context of the proportional situation.

Earl responded to the Balloon Problem as follows:

E. So they paid $2 for three balloons. You take 2 + 24.

I. Why are you doing 2 + 24?

E. To add up to see the number, because the balloons cost $2 and they want to buy twenty-four balloons for the whole class. So they need to figure how much it will cost, $26.

Clearly, Earl did not understand the multiplicative relationship that eight times as many balloons would cost eight times as much. In fact, throughout the interview, he interpreted many of the problems in terms of addition, whether or not the quantities he chose to add, such as dollars to balloons, were appropriate. For the photograph-enlargement problem (see **fig. 1**), because the 8-inch side had been enlarged by 4 inches, Earl added 4 inches to the 6-inch side instead of multiplying the length of the side by 1.5. To Earl, the enlargement was a question of adding 4 inches to each side of the photograph rather than stretching the whole photograph to 1.5 times its original size. Students like Earl and Kerry do not understand many of the fundamental components that underlie proportional reasoning.

Level 1

Level 1 strategies used by students in the interviews represent informal reasoning about proportional situations. At this level, students can think productively about problems, using manipulatives, pictures, or other models to make sense of the situations. For example, Belinda initially used a nonproportional reasoning strategy to solve the Balloon Problem by adding $2 (200 cents) and 24. She became confused when asked to explain her reasoning, and the interviewer suggested that she sketch out what she was thinking. She drew twenty-four circles on her paper, crossed out three, and wrote $2.00 (see **fig. 3a**). She continued to cross out circles in groups of three, keeping track of the $2.00 amounts on her paper. She then added the column of $2.00 amounts. Similarly, Tonya used a picture to make sense of the groups-of-students problem (see **fig. 1**). She first made five groups of five boxes (see **fig. 3b**) because "five groups with 5 divided by 25 puts five people in each group." She then labeled three boxes in each group as "G" and two boxes as "B" because "three girls were in the groups, and 2 + 3 = 5. So there are two boys in each group." She next counted the "B" boxes and correctly concluded that the class had ten boys.

Manipulatives, unit cubes in this instance, also helped students make sense of the Balloon Problem. Joycie first gave $12 as the answer. When asked by the interviewer to try the unit cubes, she made eight groups of three. She then counted by threes to verify that she had twenty-four cubes.

J. So, if three balloons cost $2, for twenty-four balloons, you need $8.

I. How did you use the cubes to determine the cost of the balloons?

J. Well, these [pointing to one group of three] cost $2, and another three will cost $4, and then another three will cost $6, $8, $10, $12, $14, $16. So it must be $16.

I. You gave a number of different answers. How can you determine which answer is correct?

J. I think $16 because I had a model show me. For $8 I was kind of guessing.

Other students started with twenty-four cubes. For example, Jody made stacks of three with her twenty-four cubes, counting by twos as she worked. Corey made similar stacks of three, then put two cubes in front of each stack to represent the $2. Then he counted the number of cubes in the two-cube stacks.

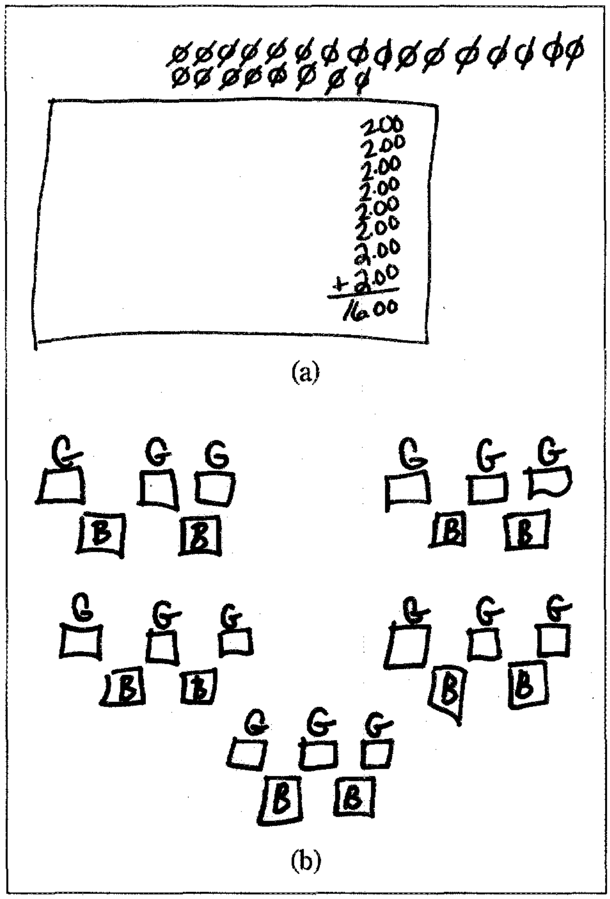


Fig. 3 **Students' informal reasoning strategies**

Both associated-sets and part-part-whole problems lend themselves to modeling with pictures, diagrams, and manipulatives. Students should be allowed to develop such strategies before being taught how to set up proportions and use the cross-product rule. This modeling helps students build on their informal reasoning to develop a better understanding of how the two measures in each of the ratios in a proportion vary together.

Level 2

At the more sophisticated strategy level 2, students can use quantitative reasoning without manipulatives or can link their models with numerical calculations. Diamond first used a nonproportional reasoning strategy for the Balloon Problem; she divided 24 by 3 to arrive at $8, then divided 24 by 2 to arrive at $12. Unable to reconcile her two answers, Diamond resorted to using the cubes and made eight groups of three.

I. So how much would it cost for twenty-four?

D. $16 because it's $2 for each pack and there is three balloons in each pack. Put each group in three, and they cost $2, so that would be eight packs that cost $16. . . . I take three in each package times eight groups equals twenty-four, and then 2 × 8 = 16. (See **fig. 4a.**)

By using cubes, Diamond understood that she would need $2 eight times. This realization enabled her to move meaningfully to the numerical calculation 2 × 8, thereby demonstrating quantitative reasoning at level 2. Other students could build up the two measures without pictures or cubes, as Tonya's solution shows (see **fig. 4b**). Marti calculated the unit price for one balloon by dividing $2 by 3 on her calculator. She got 0.66667, which she called "a wacky number," then multiplied this result by 24. Cramer and others (1993) reported that using a unit rate was students' most popular strategy and the one that was responsible for the largest number of correct answers. These researchers also found, however, that problems involving nonintegral rates, such as $0.67, were more difficult.

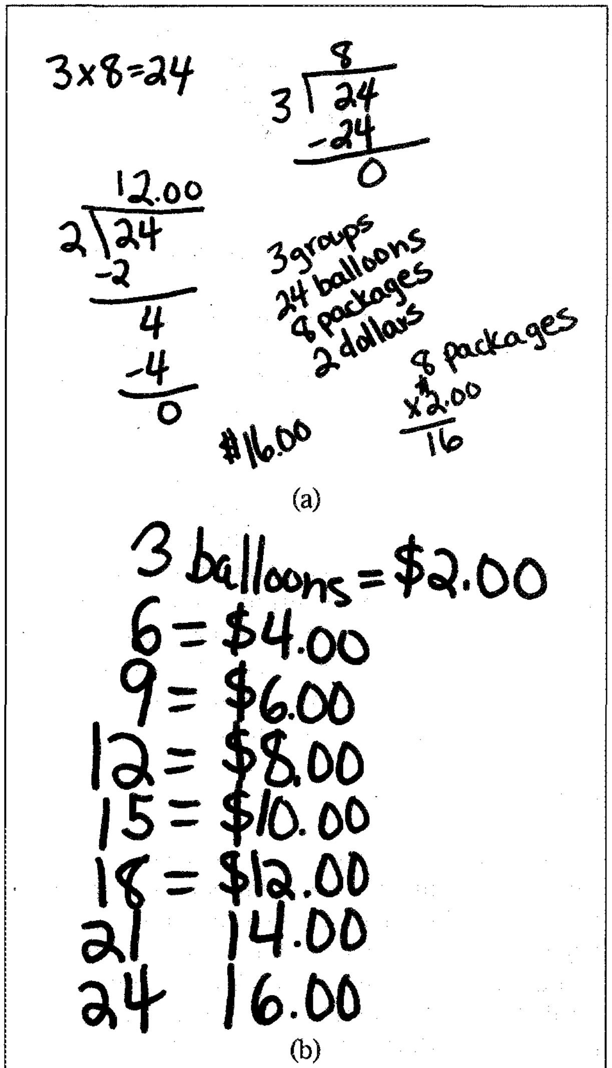


Fig. 4 Students' quantitative reasoning strategies

Level 3

At the level of formal proportional reasoning, level 3, students can set up a proportion using a variable and solve for the variable using the cross-product rule or equivalent fractions, with full understanding of the structural relationships that exist. Students must understand that the relationship between the two measures, here balloons and dollars, remains the same, that is, is *invariant,* while the two measures in each ratio vary together, that is, *covary.* In these interviews, conducted at the beginning of our work on proportional reasoning, no student demonstrated formal proportional reasoning by setting up a symbolic proportion, although setting up proportions was presented in the textbook beginning in grade 5.

Four Essential Components of Proportional Reasoning

FOUR ESSENTIAL PREREQUISITE COMPONENTS of formal proportional reasoning might help explain the limitations in our students' proportional reasoning.

Component 1

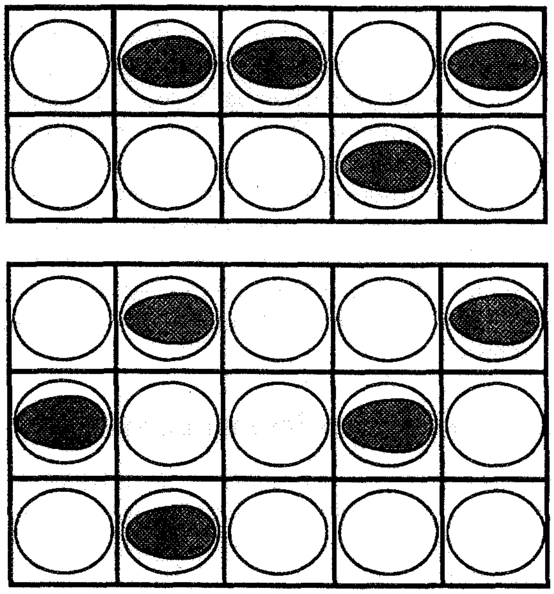
Students must recognize the difference between *absolute,* or additive, and *relative,* or multiplicative, change. Absolute change alters the original amount by an absolute, or fixed, amount, such as $10. Relative change alters the original amount by a quantity relative to the original amount, such as 10 percent. Relative change is multiplicative because the amount of the alteration is found by multiplying the original quantity by the rate, again, such as 10 percent. In the photograph-enlargement problem, Earl did not think about the change in the 8-inch side of the photograph using relative thinking. He thought in terms of the absolute change of adding 4 inches to go from 8 to 12 inches, rather than the relative change of adding one-half of the photograph's original length. Thus, he incorrectly added 4 inches to the 6-inch side, rather than 3 inches.

Teachers can help students develop relative thinking by asking, "How much?" rather than "How many?" as the problems adapted from Lamon (1995) in **figure 5** illustrate. The question "How much of each?" focuses students' attention on the part in relation to the whole rather than on an absolute quantity in and of itself.

Component 2

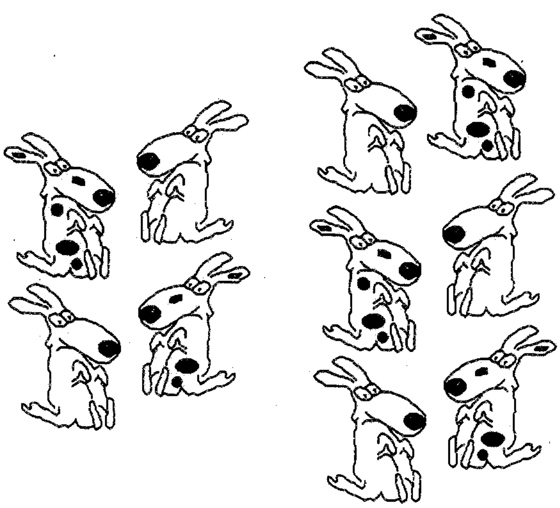
Closely related to component 1 is the need to recognize situations in which using a ratio is reasonable or appropriate. Before students begin to solve problems involving missing values in proportions, they must be able to recognize whether a ratio is the appropriate comparison. Problems such as those in **figure 6** provide this experience.

Which box of candy is nuttier? How much of each box is nuts?



(a)

Which is the most spotted litter? How much of each litter is spotted?



(b)

Adapted from Lamon (1995)

Fig. 5 How much?

Discuss the statements below. Do they make sense? What distinguishes those that make sense from those that do not?

1. If one girl can walk to school in 10 minutes, two girls can walk to school in 20 minutes.

2. If one box of cereal costs $2.80, two boxes of cereal cost $5.60.

3. If one boy makes one model car in 2 hours, then he can make three models in 6 hours.

4. If Huck can paint the fence in 2 days, then Huck, Tom, and a third boy can paint the fence in 6 days.

5. If one girl has 2 cats, then four girls have 8 cats.

When does it make sense to use a ratio?

Adapted from Lamon (1995)

Fig. 6 Making sense

Component 3

Another essential component in proportional reasoning is understanding that the quantities that make up a ratio covary in such a way that the relationship between them remains unchanged, or is *invariant.* Students tend to see problems in terms of either-or relationships; that is, either the quantities are the same or they are different. However, many different ratios can be proportional because the relationship between the two pairs of numbers is the same. Even students who can generate sets of equivalent fractions often have difficulty recognizing the invariance in equivalent ratios. The problems in **figure 7** give students the opportunity to focus on what has changed and what has remained the same. Students should also start with problems that they can solve with the help of manipulatives or pictures before proceeding to work with problems that are more difficult to model (Lamon 1995).

Component 4

The ability to build increasingly complex unit structures is essential; this approach is called *unitizing.* Students engage in qualitative proportional reasoning from level 2 when they choose one ratio as a unit and use that unit to build up to or measure the other. Diamond's use of three-packs of balloons for

During mathematics class, the fifth-grade students were grouped at three tables with 2 girls and 4 boys at each table.

During science class, they were arranged into two groups with 3 girls and 6 boys in each group.

What has changed? What has not changed?

Fig. 7 What has changed?

$2 is an example of unitizing. She used her three-pack as a unit to build up to twenty-four balloons, or eight packs. In the well-known-measures problem in **figure 1,** several students used a unit of 156 miles per 6 gallons of gasoline to determine whether some number of these mile/gallon units would equal 561 miles and 21 gallons of gasoline. Students should be presented with situations that encourage the unitizing process and prompt them to reconceptualize a whole in terms of as many different units as possible.

PUMP Instructional Changes

AFTER OUR REVIEW OF THE LITERATURE AND ASsessment of students' strategies, we shared our information with PUMP teachers through a series of seminars. We examined videotapes of the student interviews and discussed the diversity of student responses to introduce PUMP teachers to the four different problem types, students' solution strategies, and the four essential components of proportional reasoning. After gaining an understanding of the different levels of proportional reasoning, the teachers presented an associated-sets problem to their students and reported to other seminar participants the types of solution strategies exhibited in their classrooms.

These PUMP teachers refrained from introducing formal methods of setting up proportions using variables and instead asked students to represent and solve proportions informally and quantitatively using objects or pictures. We encouraged the teachers to allow their students to present different solution strategies to the class and to require their students to explain their thinking.

Together, we also examined the instructional approach to ratio and proportion that appeared in the textbook series that the PUMP teachers were using. We found that part-part-whole problems appeared in the fifth-grade textbook almost twice as often as problems involving associated sets. In the sixth-grade textbook, the problem mix between part-part-whole and associated sets was about equal. Most of the problems in seventh- and eighth-grade textbooks were growth problems, the most difficult type. Well-known-measures problems appeared at all grade levels and dominated none. Regardless of the grade level or problem type, students were asked to calculate a missing value in a proportion but never asked to make comparisons between ratios. To compensate for the deficiencies that they found, PUMP teachers supplemented the textbook with more problems of associated sets and added comparison problems to the curriculum.

Adding different problem types and encouraging and supporting students' informal and quantitative reasoning about proportion had a positive impact on student achievement. The state assessment test results showed an increase of ten points that year, although proportion problems were not specifically identified as such. Greater achievement gains are expected to occur as students spend more time over the span of the middle grades in developing and building on informal and quantitative reasoning strategies.

Recommendations for Instruction

FROM OUR WORK WITH PUMP TEACHERS AND our review of a variety of literature, we have identified several recommendations for classroom instruction.

The emphasis in textbooks is often on developing procedural skills rather than conceptual understanding. The cross-product rule is introduced early in the curriculum, without giving students an opportunity to model proportional relationships with objects or pictures. Instruction with proportional reasoning should begin with situations that can be visualized or modeled. To help students think about situations in which two measures change in relationship with each other, qualitative comparisons should be introduced before numerical comparisons are made or missing values are found. These comparisons can take the form of such statements as "If the speed is faster, you cover the same distance in less time." Beginning instruction should also emphasize informal reasoning with associated-sets and part-part-whole problems. After students can solve proportion problems using informal reasoning, the quantitative reasoning strategies of unit rates and scalar factors can be developed. Well-known-measures problems and growth problems can be introduced to those students who no longer require models. Gradually, a full range of quantitative strategies should be encouraged for solving missing-value problems. Formally setting up proportions using variables and applying the cross-product rule should be delayed until after students have had an opportunity to build on their informal knowledge and develop an understanding of the essential components of proportional reasoning.

Conclusion

PROPORTIONAL REASONING IS COMPLEX, BOTH in terms of the underlying mathematics and of the developmental experiences that it requires. Proportional reasoning must be developed over a long period of time, not in a single unit or chapter. Because proportional reasoning is used in geometry, rational numbers, and many other mathematical subject areas and because it appears to be foundational to the development of algebraic reasoning, it should be a unifying theme throughout the middle grades.

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