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Improper Applications of Proportional Reasoning

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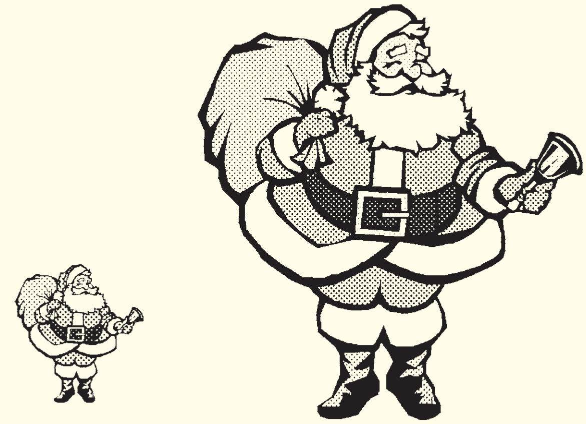
ONE OF THE MAJOR GOALS OF ELEMENTary and middle-grades mathematics education is for students to obtain a deep understanding of the proportional model in a variety of forms and applications. However, the reinforcement of proportionality at numerous occasions in school mathematics, along with the teaching of some standardized methods for solving proportionality problems, appear to lead to a resistant tendency in some students *and* adults to see and apply proportions everywhere. This same application occurs in situations where another method of solution is appropriate. Along with mastering the proportional scheme, its *misuse* seems to appear, as well. This overgeneralization of proportion has many faces: It has been found at different age levels and in a variety of mathematical domains, such as elementary arithmetic (Cramer, Post, and Currier 1993), algebra (Matz 1982), geometry (De Bock, Verschaffel, and Janssens 1998, 2002) and probability (Van Dooren et al. 2002).

De Bock, Verschaffel, and Janssens (1998, 2002) have extensively studied the tendency of 12- to 16-year-old students to improperly apply proportions in problems involving the relationship between the lengths and the area or volume in geometrically similar figures. For example, many students believe that the area of a circle with a diameter of 36 cm is three times the area of a circle with a diameter of 12 cm. In fact, in these problems a quadratic or cubic relationship is at work. Stated differently, area and volume are proportionally related to the *square* and *cube,* respectively, of the linear measures of a figure. In this article, we will illustrate how common it is for students to apply simple proportional models in this type of situation and how persistent they seem to be in this behavior.

The Broader Study

WE PERFORMED INDIVIDUAL INTERVIEWS WITH twenty 12-year-old students and twenty 16-year-old students from diverse streams of general secondary education in one school in Belgium. (The participating students were Dutch speaking. The tasks and interviews appearing in this article have been translated into English.) Each student was asked to solve a problem about the area of an enlarged irregular figure. An example is given in **figure 1.** This problem, which is about an enlarged Father Christmas (Santa Claus), cannot be solved with a simple proportion: When the Santa Claus figure is enlarged *three* times, the area (and thus the amount of paint needed) is multiplied by 9 (instead of 3). If students solved the problem in a proportional way, we gave them additional—and gradually increasing—forms of help and asked them to reconsider their own solution. In this way, we could see how persistent the students were in their use of proportions. For the sake of clarity, the hints provided in our interviews were not meant to be didactical trajectories but rather to serve to unravel students' thinking processes and test their perseverance.



Bart is a publicity painter. In the last few days, he had to paint Christmas decorations on several store windows. Yesterday, he made a drawing of a 56 cm high Father Christmas on the door of a bakery. He needed 6 ml of paint. Now he is asked to make an enlarged version of the same drawing on a supermarket window. This copy should be 168 cm high. Approximately how much paint will Bart need to do this?

Fig. 1 An example of a nonproportional problem

The findings of this study were compelling: a very strong and deep-rooted tendency was found among 12-year-old and even 16-year-old students to initially respond to the nonproportional problems with a proportional answer. They also tended to stick to the proportional model even when confronted with strong evidence that the model was incorrect for the given context. All students except two 16-year-olds initially used a simple proportion to tackle the problem. During the rest of the interview, thirty-two out of forty students abandoned the proportional model but often only after much reluctance and despair.

The two interviews that follow will give readers an idea of some of the students' problem-solving processes, the reactions observed, and the help given to students. We selected these two students because their answers represent the reactions of the entire group.

The Interview of Tommy (Aged 12)

THE RESEARCHER OFFERED TOMMY A WORKsheet containing the Santa Claus problem and the drawings shown in **figure 1.** The interview proceeded as follows:

Tommy. [Reads the problem aloud.] . . . wait, let me look for the numbers . . . . I see, the height changes from 56 cm to 168 cm. That means multiplying by 3. So, I have to multiply the paint by 3, too. [He writes down the scheme in **fig. 2.**] The answer is 18 ml. Bart will need about 18 ml for the large Santa Claus.
Interviewer. Can you elaborate a bit on your answer? Why did you solve the problem in that way?
Tommy. [Silence] Eh. I don't know. I just solved it that way.
Interviewer. Why did you multiply the amount of paint by 3?
Tommy. It is so logical. It can't be done in another way, can it? The Santa Claus becomes *higher*, so you need *more* paint. And it becomes *three times* higher, so you need *three times more* paint. It is as simple as that!
Interviewer. Can you express how sure you are that your solution is correct?
Tommy. I'm very sure. My answer is certainly correct. It is an easy problem. I just used the three numbers in the problem and the formula.

At this moment, the interviewer provided a first, rather subtle hint in the form of the table presented in **figure 3,** and continued as follows:

Interviewer. Take a look at this. . . . We gave that same problem to a large group of students in another school. This table shows how they answered it. You see: 41 percent gave the same answer as you did, namely 18 ml. But there was another 41 percent of the students who answered 54 ml. Who is right?

"You see, they made a mistake. My answer is correct"

Of course, we manipulated the table. Our idea was to show that two answers receiving equal support would raise doubts in students' minds. Moreover, students who had initially answered the problem proportionally by a lack of attention to the situation of the problem would probably recognize the correct answer once it appeared in the table. However, as Tommy's reaction illustrates, this was typically not the case.

Tommy. [Immediately] No, no, that is impossible. I have calculated it and it surely is 18 ml. How do they get 54 anyway? I will try . . . [subtracts 54 from 168; tries a few combinations of 54, 168, and 6; and tries addition, subtraction, multiplication, and division]. Oh wait, I see! They multiplied by 3 *two* times. You see, they made a mistake. My answer is correct.

Since Tommy persisted, the interviewer moved to a

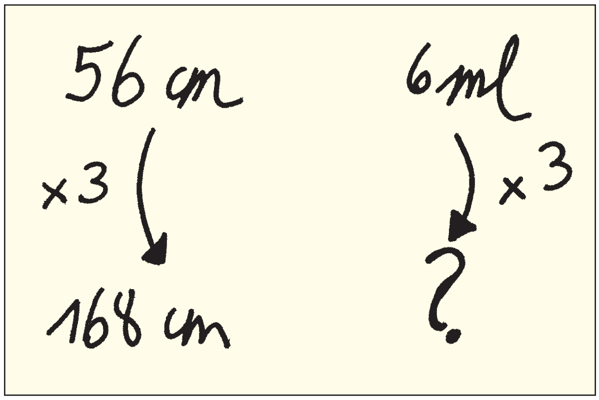


Fig. 2 The scheme written by Tommy to solve the problem

Answer	Number of Pupils
18 ml	41%
54 ml	41%
Other answers	18%

Fig. 3 A frequency table offered as a first hint, with the explanatory statement "Last week, we gave this problem to pupils in another school. . . ."

stronger kind of hint that was built into the interviewing procedure:

Interviewer. One student from that other school who answered 54 ml explained how he solved the problem. That student argued to me that if the Santa Claus picture is enlarged three times, not only the height but also the width is multiplied by 3, so that you need nine times more paint . . . .

As was the situation with the vast majority of the interviewees, Tommy was obviously not convinced by the explanation of that fictitious peer.

Tommy. Oh, that student uses the *picture*. I didn't use the picture. I just looked at the text of the problem. And the text only mentions the *height.*
Interviewer. And what if you would look at the drawing, too?
Tommy. Eighteen ml is and remains my answer. That student makes it too complex. My answer is easier.

At this point, the interviewer took another step in trying to help the student. He presented the drawings in **figure 4,** together with the following explanation:

Interviewer. Some students who answered 54 ml solved the problem in this way. They drew a rectangle around both figures and found that the figure enlarges three times in both directions: the height

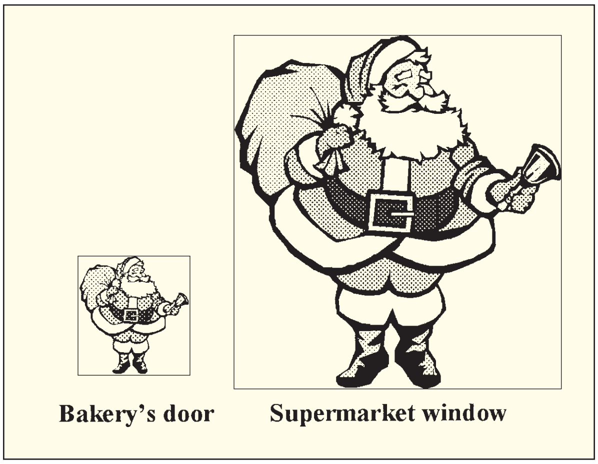


Fig. 4 These rectangles were offered as a second hint.

and the width. What do you think of that strategy? What solution do you prefer now?

This last intervention led to the following interaction with Tommy:

Tommy. But the problem says nothing about the width at all. The problem is about the height and the amount of paint.
Interviewer. And what about the strategy with the rectangles?
Tommy. What they say about the rectangles is correct: the rectangle enlarges in two directions. But within the rectangles, there is an *irregular* figure, and that’s quite different. There are white parts in the rectangles, and they are larger for the large Santa Claus. [He points at the empty spaces within the rectangles.]
Interviewer. Can you calculate the areas of the rectangles and compare them?
Tommy. [Calculates areas correctly.] This one is nine times larger.
Interviewer. And how about the Santa Claus drawings?
Tommy. [Immediately] No, that’s weird reasoning. You make it three times larger, so you need three times more paint. You make it too difficult, whereas mathematics is logical. My answer is that he needs 18 ml to paint the large Santa Claus.

At this stage, no further help was given to Tommy, and the interview was stopped. In the entire group of 12-year-olds, four students, like Tommy, persisted with their proportional answer until the end of the interview. Eight students switched to the nonproportional answer only after the last strong hint was given concerning the rectangles. The other eight 12-year-old students answered the problem correctly after we offered hints in the form of the frequency table and the explanation of the fictitious peer.

The Interview of Anne (Aged 16)

ANNE WAS ALSO GIVEN THE SANTA CLAUS PROBlem. Her interview follows.

Anne. [Reads the problem.] Oh, I see. You need 6 ml for painting 56 cm. Then I can calculate how much you need for 1 cm [divides 6 by 56 with a calculator]. Here it is! You need 0.1071 ml for 1 cm. Then I multiply by 168, because the large Santa Claus is 168 cm. [She calculates and writes the calculations shown in **fig. 5.**] You need 18 ml for the larger version of the Santa Claus.

Anne uses a strategy that differs slightly from that used by Tommy. However, her “rule of three” strategy also assumes that the amount of paint increases/decreases proportionally with the height of the figure. The interviewer worked with Anne in the same way as Tommy:

Interviewer. Can you elaborate a bit on your answer? Why did you solve the problem in that way?
Anne. It just works. I don’t know why. I do such problems always like that. You just find out how much you need for 1 cm, and the rest follows automatically.
Interviewer. How sure are you that your solution is correct?
Anne. I’m not completely sure, because I haven’t carefully read the problem. And I might have made a calculation error. But I think I did what was expected, and I used all three numbers in the problem.

Since Anne gave the proportional answer, the interviewer provided the first hint, namely, offering the frequency table **(fig. 3)** and pointing to the equally supported alternative answer. Anne reacted in this way:

Anne. What? 54 ml? I think that is quite a lot. I think my solution is more logical. Besides, it’s always better to stick to your first solution.
Interviewer. A student in that other school argued to me that if the Santa Claus picture is enlarged three times, not only the height but also the width is

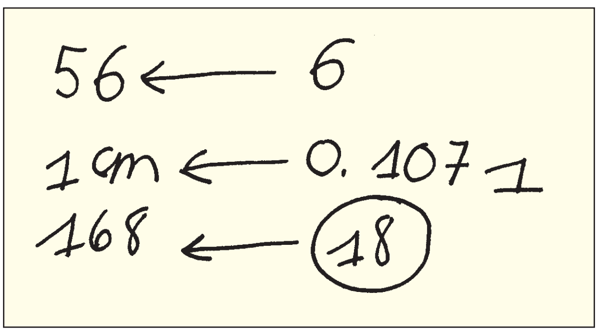


Fig. 5 Calculations made by Anne to solve the problem

multiplied by 3, so that you need nine times more paint. That's why he answered 54 ml. . . .

Anne. [Interrupts] No, I don't think so! Indeed, it becomes three times higher *and* three times wider! But that *means exactly* that the amount of paint is also multiplied by 3. The 6 ml is for the *whole* Santa Claus, not only for the height. And 18 ml is for the whole large Santa Claus. You see? This area fits three times in this area [roughly points to the small and large figure], so you need three times more paint here.

Since Anne persisted in her proportional solution—she even seemed to become more and more convinced of her answer during the interview—the next hint was provided. The interviewer showed and explained the solution strategy using rectangles (as shown in **fig. 4**). When seeing these figures, Anne immediately decided to change her answer:

Anne. Oooh yes, now I see it! It is indeed nine times larger, because the small rectangle fits nine times in the larger one. With help of these rectangles, I suddenly see it. . . . The answer is 54 ml.

Interviewer. Can you explain why you answered 18 ml at the beginning? You seemed rather convinced of your answer. . . .

Anne. Yes, but my answer seemed so logical: three times larger, three times more paint. I looked at the text and I knew immediately what I had to do. If I had taken a look at the drawings, I might have noticed that my strategy wouldn't work. But I just focused on the calculations.

Apparently, the rectangles caused a real "aha moment" in Anne's perception of the problem. Initially, she focused on the given numbers and routinely assumed that a proportional relationship existed between the lengths and the amount of paint. But when she was shown the rectangles around the irregular figures, the correct scheme suddenly became apparent: Ann realized that the area was enlarged by a factor of 9. From her last reaction, we suspect that she initially did not have a clear mental representation of the problem and, therefore, only partially applied the most prominent scheme in her repertoire, i.e., the proportional relationship.

The rectangles caused a real "aha moment" in her perception

Generally, for the twenty 16-year-olds in our study, the tendency toward improper proportional reasoning was only slightly weaker than for the 12-year-olds: two of the older students initially gave the correct answer, whereas eighteen students answered proportionally. As for the 12-year-olds, it was only after considerable help that they started to rethink their original erroneous answer and replace it with the correct one: eight students changed their answer for the nonproportional one after hearing the explanation of the fictitious peer; six others changed their answers after they saw the rectangles around the figures. Again, four students persisted in their proportional answer until the end of the interview.

Conclusions and Reflections

AS WITH THE TWO STUDENTS DESCRIBED IN THIS article, almost all students in our study spontaneously applied a solution method that assumed a proportional link between the height of a figure and its area, whereas this situation requires another mathematical model. Moreover, most of them were reluctant to reconsider their initial solution or had serious difficulties in understanding and appreciating an alternative model, even after several increasing forms of help.

This interview study enabled us to get an idea of the actual problem-solving processes taking place when students incorrectly solve problems in a proportional way and of the elements in their knowledge base that can explain their adherence to that proportional solution, such as inadequate habits and particular shortcomings in their knowledge of geometry. One major influencing factor seems to be the formulation of problems in a missing-value format (wherein three numbers are given and a fourth is to be determined). With this linguistic format, students learned almost automatically to apply a proportional scheme and solution method throughout their school career (see, e.g., Cramer and Post 1993). By applying the solution scheme that they were most familiar with, the students in our study appeared not to be aware of the quadratic growth of the area when a figure is enlarged. An important conclusion of our research is that paying too much attention to specific methods of solution for proportional problems without also focusing on the underlying concepts and relationships of a problem situation may well turn out to be counterproductive. When students develop a deep conceptual understanding of proportionality, they should also develop the disposition to distinguish between situations that can and cannot be modeled proportionally.

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