Improving Middle School Teachers' Reasoning about Proportional Reasoning

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A GROUP OF MIDDLE AND ELEMENTARY school teachers participating in a graduate mathematics methods course was given the following problem:

> Sue and Julie are running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run? (Lamon 1994)

Below is an excerpt from the conversation of one group as they worked on the problem:

Jafir. Oh, this is easy. Let's just set up a proportion and solve it.

Emily. Yeah, let's see. It's $9/3 = x/15$. Cross multiplying, we get $3x = 135$, so $x = 45$.

Jesse. It's a whole number, so it must be right. This proportion stuff is really easy—nothing to it. My kids get this cross-multiplying rule pretty quickly. Once they get the numbers set up, it's a piece of cake.

Jafir. Yeah, mine, too. It's easy for them as long as the numbers come out nice and neat. They have problems when fractions are involved.

Denille [who has been reading and working on the problem quietly]. You know, I don't think that answer is right. It really doesn't make much sense when you draw a picture of a running track and mentally move the girls around the track.

Jesse. It has to be right. It's a proportion problem; we set up the proportion and did the computation correctly. It's just like the others we have been doing in class.

Emily. No, it isn't. Denille is right. This one is different. You just subtract, then add. Just think about it for a minute.

The Importance of Proportional Reasoning

"THE ABILITY TO REASON PROPORTIONALLY develops in students throughout grades 5–8. It is of such great importance that it merits whatever time and effort must be expended to assure its careful development. Students need to see many problem situations that can be modeled and then solved through proportional reasoning" (NCTM 1989, p. 82). Proportional reasoning pervades the middle school curriculum. The authors of *Principles and Standards for School Mathematics* identify the following topics in the middle school mathematics curriculum as ones involving proportional reasoning: ratio, percent, similarity, scaling, linear equations, slope, relative-frequency histograms, and probability (NCTM 2000, p. 212). Proportional reasoning also pervades the secondary school mathematics curriculum and is essential in the study of linear equations, rates, rational numbers and expressions, and similar figures and their area and volume relationships. Furthermore, proportional reasoning is necessary for solving problems in all branches of science. Proportional reasoning is the capstone of the elementary school curriculum and the cornerstone of high school mathematics and science (Post, Behr, and Lesh 1988).

For middle school teachers to help their students become proportional reasoners, they must understand the characteristics of proportional reasoning and know that it is developmental, emerges gradually, and grows over a span of several years. Teaching students to solve proportions by using the cross-product method alone does not develop the students' proportional reasoning skills. Proportional reasoning is a way of thinking, not an algorithm to be used in solving problems. To facilitate students' development of proportional reasoning, teachers must be proportional reasoners themselves and be able to determine when their students are capable of reasoning proportionally.

A group of middle school teachers in Kentucky recognized the need to increase their knowledge of proportional reasoning. As part of a statewide professional development project, teachers in six of eight regions selected proportional reasoning as one of three important mathematics topics to study in summer academies.

Kentucky Middle School Mathematics Academies

A PROFESSIONAL DEVELOPMENT MODULE focusing on proportional reasoning was included as part of the Kentucky Middle School Mathematics Academies to upgrade the content knowledge of middle school mathematics teachers. Approximately 300 teachers participated in a week of professional development activities for three summers and for an additional twelve hours of follow-up activities during subsequent academic years.

Building a Proportional Reasoning Module

A TEAM OF FIVE MATHEMATICS EDUCATORS developed the proportional reasoning module that was used in the academy. The team included two mathematics education professors, a middle school mathematics teacher, and two former middle school teachers who now serve as regional mathematics consultants for the Kentucky Department of Education. In planning the module, the team conducted a thorough examination of the role of proportional reasoning in middle and secondary school mathematics curricula and reviewed research about proportional reasoning. The team then determined the following objectives, which specified that after completing the module, teachers should be able to—

- distinguish between proportional and nonproportional reasoning;
- demonstrate proportional reasoning in problem-solving situations;
- explain the importance and pervasiveness of proportional reasoning in real-world situations;
- identify the characteristics of proportional reasoning;
- assess proportional reasoning abilities in middle school students;
- communicate research results related to proportional reasoning;
- identify proportional reasoning tasks across the mathematics curriculum; and
- solve proportions with and without using the cross-product method.

The team then reviewed activities from a wide variety of sources and created others that aligned with *Principles and Standards* (NCTM 2000) and Kentucky's middle school core mathematics curriculum (Kentucky Department of Education 1999). In addition, the team administered short proportional reasoning tasks to about 300 students in grades 5–8 in school districts across the state. The team wanted to obtain success rates and thinking patterns of proportional reasoners and nonproportional reasoners to share and analyze the students' work with middle school teachers in the academies.

The resulting two-day module included eighteen activities. Some were variations of activities found elsewhere, and others were original activities that team members had used with students or teachers. Each activity in the module was described fully; activity materials included the source of the activity, objectives, materials needed and time required, necessary preparations, step-by-step procedures, discussion issues and questions, and worksheets and transparency masters. Two teacher assessments were also developed, one to be used midway through the module and the other, at the end. These assessments consisted of open-ended questions that focused on the type of thinking required of proportional reasoners, as well as on applications of proportional reasoning.

Sharing the Proportional Reasoning Module

REGIONAL TEAMS OF MATHEMATICS EDUCATORS shared the proportional reasoning module with six groups of twenty to thirty middle school teachers across the state of Kentucky. The professional development module was presented during half of a five-day academy session, along with a module on another topic. Activities in the proportional reasoning module were mainly of two types. One type involved relatively easy proportional reasoning tasks using simple numbers or no numbers. These tasks were used to focus on the characteristics of proportional and nonproportional thinking. The other activities were applications of proportional reasoning across the middle-grades mathematics curriculum. These activities typically involved the use of hands-on learning, data collection and analysis in small groups, and calculators or computer-based spreadsheets to solve problems simulating real-world situations.

Proportional Reasoning Activities with Simple Numbers or No Numbers

THE PROPORTIONAL REASONING TASKS GIVEN TO approximately 300 middle school students across the state during the previous school year were central to the module. The participating teachers were asked to solve each task individually, then to share their reasoning with partners. The following task, Sam the Snake, yielded interesting discussions:

> Sam the snake is 4 feet long. When he is fully grown, he will be 8 feet long. Sally the snake is 5 feet long. When she is fully grown, she will be 9 feet long. Which snake is closer to being fully grown? Explain how you know. (Lamon 1994)

As teachers shared their reasoning, team members made distinctions between thinking that involved multiplicative reasoning, the key component of proportional reasoning, and thinking that involved additive reasoning, a precursor to multiplicative reasoning. After discussions about the teachers' thinking, team members shared examples of students' thinking that had been gathered previously:

> *Proportional thinking:* "Sally the snake is closer to being fully grown because she is 5 feet long and will only get to 9 feet. This means she is over half of her total growth. Although Sam will only get to be 8 feet, he is exactly halfway to his total growth, and this means Sally is further along in growth than Sam." This response shows multiplicative reasoning because the student notes that Sam's final length is twice the intermediate length, but Sally's final length is less than twice the intermediate length.

> *Nonproportional thinking:* "They are both the same distance to being fully grown. Although Sam is only 4 feet long and will be 8 feet long, Sally is 5 feet long and will be 9 feet. No matter how you look at it, they are both 4 feet away from being fully grown." This response shows additive reasoning because the student believes that the correct calculation for Sam is $8 = 4 + 4$ and, for Sally, $9 = 5 + 4$.

Success rates for the sample of middle school students solving the task were shared, revealing the percentages of students in each of grades 5–8 who solved the problem by using proportional reasoning. The success rates on this task were 11 percent for fifth graders, 27 percent for sixth graders, 40 percent for seventh graders, and 52 percent for eighth graders. Team members then led a discussion about the developmental nature of proportional reasoning, the overall performance on the task, the need for teachers to help middle school students develop their proportional reasoning abilities, and the use of such tasks as Sam the Snake to explore the differences between multiplicative thinking and additive thinking.

Qualitative tasks also were used to explore the characteristics of proportional reasoning. These tasks included no numbers and provide interesting insights into student thinking. One such task was the following from Cramer and Post (1992):

> Two friends mix blue tint with white paint to make some blue paint. Decide which friend mixed the darkest shade of blue paint. Nancy used more blue tint than Kathy. Nancy mixed in more white paint than Kathy. Who mixed the darkest shade of blue?
>
> a) Nancy
> b) Kathy
> c) Their paint mixtures were exactly the same.
> d) There is not enough information to tell.

After the teachers discussed their reasoning related to this task, the team members shared the following examples of children's thinking:

> *Proportional thinker:* "The answer is (d) because you need to know how much blue tint and how much white paint each mixed to get the answer. It only tells that Nancy mixed more blue tint and more white paint than Kathy. This only tells me that Nancy will have more paint. You need to know how many units of blue tint were used for every unit of white paint."

> *Nonproportional thinker 1:* "Nancy because she put in more tint of blue, and the more you put in it, the darker it gets."

> *Nonproportional thinker 2:* "Kathy because the more blue tint you add, the darker your color will become. Also, Nancy mixes in more white paint, so that would make her color lighter."

Nonproportional thinker 3: "Their paint mixtures were exactly the same. The paint mixtures are the same because even though it said Nancy used more blue tint than Kathy, it also says Nancy mixed in more white paint than Kathy. This would make the mixtures seem the same."

In discussing this task and others with teachers in these professional development sessions, team members had teachers share their thinking and solution methods without making any judgments about the correctness of the teachers' solutions or thinking patterns. This approach allowed everyone to explore the relationships involved in the problems and further develop their abilities to think proportionally. By reflecting on their own reasoning, as well as on samples of students' reasoning, teachers (a) came to a better understanding of their own thinking, including their sometimes erroneous thinking; (b) learned the fundamental differences in proportional and nonproportional reasoning; and (c) became aware of the power of simple proportional reasoning tasks in stimulating students' thinking.

Proportional Reasoning Activities across the Middle School Mathematics Curriculum

ANOTHER TYPE OF ACTIVITY IN THE MODULE required teachers to solve proportional reasoning problems from one or more of the following mathematics content areas: number and computation, geometry and measurement, probability and statistics, and algebraic reasoning. The purpose of these activities was to demonstrate the pervasiveness of proportional reasoning in the middle school mathematics curriculum and to provide teachers with the opportunity to apply proportional reasoning in these contexts. The following activity was closely linked to measurement:

> You are the ace detective for a local law enforcement agency. You have been called in to help solve a case. Suspects have been narrowed to 3 people. One is 5 feet tall. The second is 6 feet tall. The third is 7 feet tall. The only clue left from the scene of the crime is a handprint. Use the handprint to help narrow the list of suspects. [A handprint that had a span approximately one-third greater than that of a typical adult who is 6 feet tall was shown with the problem.]

Teachers discussed possible solution methods in small groups and shared them with the entire class. This discussion usually led to ideas about collecting data from the class members about their hand spans and related heights. The data were collected and entered into graphing calculators or spreadsheets structured as shown in **figure 1.** Teachers then analyzed the data and decided how to use the ratios in column C along with the span of the distributed handprint to predict the height of the suspect. Some groups decided to find and use the mean of the ratios in column C, and others decided to use the median of those ratios. The teachers then found the predicted height of the suspect by multiplying the span of the handprint by the mean or median of the ratios.

	A	B	C
1	HEIGHT	HAND SPAN	RATIO
2	<value>	<value>	=A2/B2
3			

Fig. 1 Spreadsheet format for detective activity

The team led a discussion that focused on how proportional reasoning was involved in this problem. Usually, comments centered on the idea that the mean or median ratio, height/hand span, represented the person's height in terms of his or her hand span; such comments reveal multiplicative reasoning. The predicted height of the suspect could, therefore, be found by multiplying the mean or median ratio and the hand span of the suspect. Further discussion focused on the use of proportional reasoning in the profession of forensic medicine.

Without a doubt, one of the most intriguing and provocative activities in the module was "What Time Is It?" (Gallin 1999) (see **fig. 2**). For this task, the teachers were shown a picture of a clock face having both an hour hand and a minute hand set in specific positions. The intriguing part of the task was that the clock face had no numerals, and the 12 did not necessarily belong at the top.

The task required teachers to determine the correct time by using the relationship between the minute

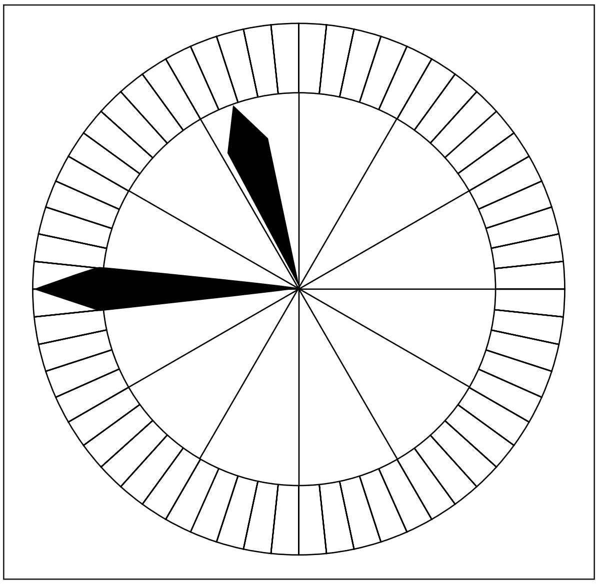

Fig. 2 "What Time Is It?" (From Gallin [1999])

hand and the portion of an hour that it had traveled around the clock face, along with the hour hand and the portion of an hour that it had traveled from one of the hour marks on the clock to the next hour mark. In fact, these two relationships could be represented as equal ratios—one of the most frequently accepted definitions of proportional reasoning.

After teachers worked on the problem and shared their thinking, several ideas became clear. The most common solution method was to rotate the clock to various positions, mentally place 12 at the top each time, determine what fraction of the total trip around the clock the minute hand had completed, and check to see if the hour hand had completed the same fraction of the trip between the two hour numbers on both sides of it. A number of teachers were surprised to learn that the movement of the hour hand was so precise, but all were excited about the task and thought that it would be interesting, challenging, and motivating for their students.

In the context of each of these problems, the team members and the teachers discussed the differences between solving proportions and thinking proportionally. All agreed that proportional reasoning is not a well-developed curricular topic and that attention is normally given to solving proportions procedurally by using cross products. The teachers explored other methods of solving simple proportions by using proportional reasoning, such as working with unit rates and factor of change.

Follow-up Sessions during the Subsequent School Year

DURING THE SUBSEQUENT SCHOOL YEAR, THE team members held a six-hour professional development session addressing proportional reasoning. At this session, the team focused on experiences of the teachers when they taught proportional reasoning in their classrooms. Each teacher brought an activity using proportional reasoning to share, along with students' work samples. This sharing opportunity led to spirited discussions, and teachers were keenly interested in what others had done. The team also presented new activities involving proportional reasoning. One of the most interesting of these activities used elastic bands, which can be obtained at a sewing store, each marked by the user into 100 equal subdivisions to model percents and decimals. By stretching a marked band to match the length of a person's arm from shoulder to fingertip, for example, the user can see that the hand is about 25 percent of the arm's length. In effect, the bands model the meaning of *percent;* they divide an object into 100 parts to help the user determine the number of parts (the percent) corresponding to a certain component of the object (AIMS Education Foundation 2000).

Outcomes

ASSESSMENTS REVEALED THAT TEACHERS IMproved their own proportional reasoning skills, they learned more about student thinking, and they learned important applications that require proportional reasoning. One teacher's comments during the subsequent school year serve as testimony to the value of the professional development sessions: "My work with the students, after using some specific proportional reasoning activities from the workshop, yielded much better products. The students were probing and looking for their own meaning, as opposed to simply looking for the right answer. And, to my surprise, I found that many of them were not satisfied with just the right answer but wanted to share how they got there."

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